search words. Since the list of mathematics search words does not contain any other dental terms, a mathematics search using an approximate database matrix is likely to eliminate all documents relating to dentistry. Similarly, the mathematics documents would be filtered out in the dental student's search.

Web Searches and Page Ranking

Modern Web searches could easily involve billions of documents with hundreds of thousands of keywords. Indeed, as of July 2008, there were more than 1 trillion Web pages on the Internet, and it is not uncommon for search engines to acquire or update as many as 10 million Web pages in a single day. Although the database matrix for pages on the Internet is extremely large, searches can be simplified dramatically, since the matrices and search vectors are *sparse*; that is, most of the entries in any column are 0's.

For Internet searches, the better search engines will do simple matching searches to find all pages matching the keywords, but they will not order them on the basis of the relative frequencies of the keywords. Because of the commercial nature of the Internet, people who want to sell products may deliberately make repeated use of keywords to ensure that their Web site is highly ranked in any relative-frequency search. In fact, it is easy to surreptitiously list a keyword hundreds of times. If the font color of the word matches the background color of the page, then the viewer will not be aware that the word is listed repeatedly.

For Web searches, a more sophisticated algorithm is necessary for ranking the pages that contain all of the key search words. This type of model is referred to as a *Markov process* or a *Markov chain.*

References

1. Berry, Michael W., and Murray Browne, *Understanding Search Engines: Mathematical Modeling and Text Retrieval*, SIAM, Philadelphia, 1999.

SECTION 3 EXERCISES

1. If

$$A = \begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{pmatrix}$$

compute

(a) $2A$ **(b)** $A + B$

(c) $2A - 3B$ **(d)** $(2A)^T - (3B)^T$

(e) AB **(f)** BA

(g) $A^T B^T$ **(h)** $(BA)^T$

2. For each of the pairs of matrices that follow, determine whether it is possible to multiply the first matrix times the second. If it is possible, perform the multiplication.

(a) $\begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -2 \\ 6 & -4 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & 6 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 6 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{pmatrix}$

(f) $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 4 & 5 \end{pmatrix}$

3. For which of the pairs in Exercise 2 is it possible to multiply the second matrix times the first, and what would the dimension of the product matrix be?

4. Write each of the following systems of equations as a matrix equation.

(a) $3x_1 + 2x_2 = 1$
$2x_1 - 3x_2 = 5$

(b) $x_1 + x_2 = 5$
$2x_1 + x_2 - x_3 = 6$
$3x_1 - 2x_2 + 2x_3 = 7$

(c) $2x_1 + x_2 + x_3 = 4$
$x_1 - x_2 + 2x_3 = 2$
$3x_1 - 2x_2 - x_3 = 0$

5. If

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 7 \end{pmatrix}$$

verify that

(a) $5A = 3A + 2A$ **(b)** $6A = 3(2A)$
(c) $(A^T)^T = A$

6. If

$$A = \begin{pmatrix} 4 & 1 & 6 \\ 2 & 3 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{pmatrix}$$

verify that

(a) $A + B = B + A$
(b) $3(A + B) = 3A + 3B$
(c) $(A + B)^T = A^T + B^T$

7. If

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ -2 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix}$$

verify that

(a) $3(AB) = (3A)B = A(3B)$
(b) $(AB)^T = B^T A^T$

8. If

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} -2 & 1 \\ 0 & 4 \end{pmatrix}, C = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

verify that

(a) $(A + B) + C = A + (B + C)$
(b) $(AB)C = A(BC)$
(c) $A(B + C) = AB + AC$
(d) $(A + B)C = AC + BC$

9. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

(a) Write $\mathbf{b}$ as a linear combination of the column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.
(b) Use the result from part (a) to determine a solution of the linear system $A\mathbf{x} = \mathbf{b}$. Does the system have any other solutions? Explain.
(c) Write $\mathbf{c}$ as a linear combination of the column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

10. For each of the choices of A and $\mathbf{b}$ that follow, determine whether the system $A\mathbf{x} = \mathbf{b}$ is consistent by examining how $\mathbf{b}$ relates to the column vectors of A. Explain your answers in each case.

(a) $A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

11. Let A be a 5×3 matrix. If

$$\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_2 + \mathbf{a}_3$$

then what can you conclude about the number of solutions of the linear system $A\mathbf{x} = \mathbf{b}$? Explain.

12. Let A be a 3×4 matrix. If

$$\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4$$

then what can you conclude about the number of solutions of the linear system $A\mathbf{x} = \mathbf{b}$? Explain.

13. Let $A\mathbf{x} = \mathbf{b}$ be a linear system whose augmented matrix $(A|\mathbf{b})$ has reduced row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(a) Find all solutions to the system.

(b) If

$$\mathbf{a}_1 = \begin{Bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{Bmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{Bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{Bmatrix}$$

determine **b**.

14. Let A be an $m \times n$ matrix. Explain why the matrix multiplications A^TA and AA^T are possible.

15. A matrix A is said to be *skew symmetric* if $A^T = -A$. Show that if a matrix is skew symmetric, then its diagonal entries must all be 0.

16. In Application 2, suppose that we are searching the database of seven linear algebra books for the search words *elementary, matrix, algebra.* Form a search vector **x**, and then compute a vector **y** that represents the results of the search. Explain the significance of the entries of the vector **y**.

17. Let A be a 2×2 matrix with $a_{11} \neq 0$ and let $\alpha = a_{21}/a_{11}$. Show that A can be factored into a product of the form

$$\begin{Bmatrix} 1 & 0 \\ \alpha & 1 \end{Bmatrix} \begin{Bmatrix} a_{11} & a_{12} \\ 0 & b \end{Bmatrix}$$

What is the value of b?

4 Matrix Algebra

The algebraic rules used for real numbers may or may not work when matrices are used. For example, if a and b are real numbers then

$$a + b = b + a \quad \text{and} \quad ab = ba$$

For real numbers, the operations of addition and multiplication are both commutative. The first of these algebraic rules works when we replace a and b by square matrices A and B; that is,

$$A + B = B + A$$

However, we have already seen that matrix multiplication is not commutative. This fact deserves special emphasis.

> **Warning:** In general, $AB \neq BA$. Matrix multiplication is *not* commutative.

In this section we examine which algebraic rules work for matrices and which do not.

Algebraic Rules

The following theorem provides some useful rules for doing matrix algebra:

Theorem 4.1 Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$