SECTION 4 EXERCISES

1. Explain why each of the following algebraic rules will not work in general when the real numbers a and b are replaced by $n \times n$ matrices A and B.
   (a) $(a+b)^2 = a^2 + 2ab + b^2$
   (b) $(a+b)(a-b) = a^2 - b^2$

2. Will the rules in Exercise 1 work if a is replaced by an $n \times n$ matrix A and b is replaced by the $n \times n$ identity matrix I?

3. Find nonzero 2×2 matrices A and B such that $AB = O$.

4. Find nonzero matrices A, B, and C such that
$$AC = BC \quad \text{and} \quad A \neq B$$

5. The matrix
$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$
has the property that $A^2 = O$. Is it possible for a nonzero symmetric 2×2 matrix to have this property? Prove your answer.

6. Prove the associative law of multiplication for 2×2 matrices; that is, let
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$
$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$
and show that
$$(AB)C = A(BC)$$

7. Let
$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
Compute A^2 and A^3. What will A^n turn out to be?

8. Let
$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
Compute A^2 and A^3. What will A^{2n} and A^{2n+1} turn out to be?

9. Let
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Show that $A^n = O$ for $n \geq 4$.

10. Let A and B be symmetric $n \times n$ matrices. For each of the following, determine whether the given matrix must be symmetric or could be nonsymmetric:
    (a) $C = A + B$ (b) $D = A^2$
    (c) $E = AB$ (d) $F = ABA$
    (e) $G = AB + BA$ (f) $H = AB - BA$

11. Let C be a nonsymmetric $n \times n$ matrix. For each of the following, determine whether the given matrix must be symmetric or could be nonsymmetric:
    (a) $A = C + C^T$ (b) $B = C - C^T$
    (c) $D = C^TC$ (d) $E = C^TC - CC^T$
    (e) $F = (I + C)(I + C^T)$
    (f) $G = (I + C)(I - C^T)$

12. Let
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then
$$A^{-1} = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

13. Use the result from Exercise 12 to find the inverse of each of the following matrices:
    (a) $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

14. Let A and B be $n \times n$ matrices. Show that if
$$AB = A \quad \text{and} \quad B \neq I$$
then A must be singular.

15. Let A be a nonsingular matrix. Show that A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$.

16. Prove that if A is nonsingular, then A^T is nonsingular and
$$(A^T)^{-1} = (A^{-1})^T$$
[*Hint*: $(AB)^T = B^TA^T$.]

17. Let A be an $n \times n$ matrix and let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Show that if $A\mathbf{x} = A\mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then the matrix A must be singular.

18. Let A be a nonsingular $n \times n$ matrix. Use mathematical induction to prove that A^m is nonsingular and

$$(A^m)^{-1} = (A^{-1})^m$$

for $m = 1, 2, 3, \ldots$.

19. Let A be an $n \times n$ matrix. Show that if $A^2 = O$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

20. Let A be an $n \times n$ matrix. Show that if $A^{k+1} = O$, then $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^k$$

21. Given

$$R = \begin{Bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{Bmatrix}$$

show that R is nonsingular and $R^{-1} = R^T$.

22. An $n \times n$ matrix A is said to be an *involution* if $A^2 = I$. Show that if G is any matrix of the form

$$G = \begin{Bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{Bmatrix}$$

then G is an involution.

23. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ (i.e., $\mathbf{u}^T\mathbf{u} = 1$) and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is an involution.

24. A matrix A is said to be *idempotent* if $A^2 = A$. Show that each of the following matrices are idempotent:

(a) $\begin{Bmatrix} 1 & 0 \\ 1 & 0 \end{Bmatrix}$ **(b)** $\begin{Bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{Bmatrix}$

(c) $\begin{Bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix}$

25. Let A be an idempotent matrix.

(a) Show that $I - A$ is also idempotent.

(b) Show that $I + A$ is nonsingular and $(I + A)^{-1} = I - \frac{1}{2}A$

26. Let D be an $n \times n$ diagonal matrix whose diagonal entries are either 0 or 1.

(a) Show that D is idempotent.

(b) Show that if X is a nonsingular matrix and $A = XDX^{-1}$, then A is idempotent.

27. Let A be an involution matrix, and let

$$B = \frac{1}{2}(I + A) \quad \text{and} \quad C = \frac{1}{2}(I - A)$$

Show that B and C are both idempotent and $BC = O$.

28. Let A be an $m \times n$ matrix. Show that A^TA and AA^T are both symmetric.

29. Let A and B be symmetric $n \times n$ matrices. Prove that $AB = BA$ if and only if AB is also symmetric.

30. Let A be an $n \times n$ matrix and let

$$B = A + A^T \quad \text{and} \quad C = A - A^T$$

(a) Show that B is symmetric and C is skew symmetric.

(b) Show that every $n \times n$ matrix can be represented as a sum of a symmetric matrix and a skew-symmetric matrix.

31. In Application 1, how many married women and how many single women will there be after 3 years?

32. Consider the matrix

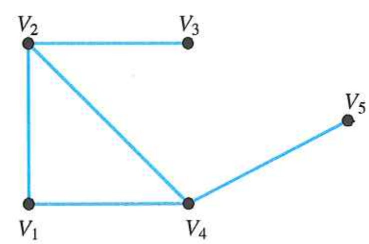
$$A = \begin{Bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{Bmatrix}$$

(a) Draw a graph that has A as its adjacency matrix. Be sure to label the vertices of the graph.

(b) By inspecting the graph, determine the number of walks of length 2 from V_2 to V_3 and from V_2 to V_5.

(c) Compute the second row of A^3, and use it to determine the number of walks of length 3 from V_2 to V_3 and from V_2 to V_5.

33. Consider the graph



(a) Determine the adjacency matrix A of the graph.

(b) Compute A^2. What do the entries in the first row of A^2 tell you about walks of length 2 that start from V_1?

(c) Compute A^3. How many walks of length 3 are there from V_2 to V_4? How many walks of length less than or equal to 3 are there from V_2 to V_4?

For each of the conditional statements that follow, answer true *if the statement is always true and answer* false *otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true.*

34. If $A\mathbf{x} = B\mathbf{x}$ for some nonzero vector $\mathbf{x}$, then the matrices A and B must be equal.

35. If A and B are singular $n \times n$ matrices, then $A + B$ is also singular.

36. If A and B are nonsingular matrices, then $(AB)^T$ is nonsingular and

$$((AB)^T)^{-1} = (A^{-1})^T(B^{-1})^T$$

5 Elementary Matrices

In this section, we view the process of solving a linear system in terms of matrix multiplications rather than row operations. Given a linear system $A\mathbf{x} = \mathbf{b}$, we can multiply both sides by a sequence of special matrices to obtain an equivalent system in row echelon form. The special matrices we will use are called *elementary matrices*. We will use them to see how to compute the inverse of a nonsingular matrix and also to obtain an important matrix factorization. We begin by considering the effects of multiplying both sides of a linear system by a nonsingular matrix.

Equivalent Systems

Given an $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, we can obtain an equivalent system by multiplying both sides of the equation by a nonsingular $m \times m$ matrix M:

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

$$MA\mathbf{x} = M\mathbf{b} \tag{2}$$

Clearly, any solution of (1) will also be a solution of (2). On the other hand, if $\hat{\mathbf{x}}$ is a solution of (2), then

$$M^{-1}(MA\hat{\mathbf{x}}) = M^{-1}(M\mathbf{b})$$

$$A\hat{\mathbf{x}} = \mathbf{b}$$

and it follows that the two systems are equivalent.

To transform the system $A\mathbf{x} = \mathbf{b}$ to a simpler form that is easier to solve, we can apply a sequence of nonsingular matrices $E_1, \ldots, E_k$ to both sides of the equation. The new system will then be of the form

$$U\mathbf{x} = \mathbf{c}$$

where $U = E_k \cdots E_1 A$ and $\mathbf{c} = E_k \cdots E_2 E_1 \mathbf{b}$. The transformed system will be equivalent to the original, provided that $M = E_k \cdots E_1$ is nonsingular. However, M is nonsingular, since it is a product of nonsingular matrices.

We will show next that any of the three elementary row operations can be accomplished by multiplying A on the left by a nonsingular matrix.

Elementary Matrices

If we start with the identity matrix I and then perform exactly one elementary row operation, the resulting matrix is called an *elementary* matrix.

There are three types of elementary matrices corresponding to the three types of elementary row operations.