SECTION 5 EXERCISES

1. Which of the matrices that follow are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$ **(b)** $\begin{Bmatrix} 2 & 0 \\ 0 & 3 \end{Bmatrix}$

(c) $\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{Bmatrix}$ **(d)** $\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$

2. Find the inverse of each matrix in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

3. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$:

(a) $A = \begin{Bmatrix} 2 & -1 \\ 5 & 3 \end{Bmatrix}$, $B = \begin{Bmatrix} -4 & 2 \\ 5 & 3 \end{Bmatrix}$

(b) $A = \begin{Bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{Bmatrix}$, $B = \begin{Bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{Bmatrix}$

(c) $A = \begin{Bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{Bmatrix}$,

$B = \begin{Bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{Bmatrix}$

4. For each of the following pairs of matrices, find an elementary matrix E such that $AE = B$:

(a) $A = \begin{Bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{Bmatrix}$, $B = \begin{Bmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{Bmatrix}$

(b) $A = \begin{Bmatrix} 2 & 4 \\ 1 & 6 \end{Bmatrix}$, $B = \begin{Bmatrix} 2 & -2 \\ 1 & 3 \end{Bmatrix}$

(c) $A = \begin{Bmatrix} 4 & -2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{Bmatrix}$,

$B = \begin{Bmatrix} 2 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{Bmatrix}$

5. Let

$$A = \begin{Bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{Bmatrix}, \quad B = \begin{Bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{Bmatrix},$$

$$C = \begin{Bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{Bmatrix}$$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $FB = C$.

(c) Is C row equivalent to A? Explain.

6. Let

$$A = \begin{Bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{Bmatrix}$$

(a) Find elementary matrices E_1, E_2, E_3 such that

$$E_3 E_2 E_1 A = U$$

where U is an upper triangular matrix.

(b) Determine the inverses of E_1, E_2, E_3 and set $L = E_1^{-1} E_2^{-1} E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

7. Let

$$A = \begin{Bmatrix} 2 & 1 \\ 6 & 4 \end{Bmatrix}$$

(a) Express A as a product of elementary matrices.

(b) Express A^{-1} as a product of elementary matrices.

8. Compute the LU factorization of each of the following matrices:

(a) $\begin{Bmatrix} 3 & 1 \\ 9 & 5 \end{Bmatrix}$ **(b)** $\begin{Bmatrix} 2 & 4 \\ -2 & 1 \end{Bmatrix}$

(c) $\begin{Bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{Bmatrix}$ **(d)** $\begin{Bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{Bmatrix}$

9. Let

$$A = \begin{Bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{Bmatrix}$$

(a) Verify that

$$A^{-1} = \begin{Bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{Bmatrix}$$

(b) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ for the following choices of $\mathbf{b}$:

(i) $\mathbf{b} = (1, 1, 1)^T$ **(ii)** $\mathbf{b} = (1, 2, 3)^T$
(iii) $\mathbf{b} = (-2, 1, 0)^T$

10. Find the inverse of each of the following matrices:

(a) $\begin{Bmatrix} -1 & 1 \\ 1 & 0 \end{Bmatrix}$ **(b)** $\begin{Bmatrix} 2 & 5 \\ 1 & 3 \end{Bmatrix}$

(c) $\begin{Bmatrix} 2 & 6 \\ 3 & 8 \end{Bmatrix}$ **(d)** $\begin{Bmatrix} 3 & 0 \\ 9 & 3 \end{Bmatrix}$

(e) $\begin{Bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{Bmatrix}$ **(f)** $\begin{Bmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{Bmatrix}$

(g) $\begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

11. Given

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

compute A^{-1} and use it to

(a) find a 2×2 matrix X such that $AX = B$.

(b) find a 2×2 matrix Y such that $YA = B$.

12. Let

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

Solve each of the following matrix equations:

(a) $AX + B = C$ **(b)** $XA + B = C$

(c) $AX + B = X$ **(d)** $XA + C = X$

13. Is the transpose of an elementary matrix an elementary matrix of the same type? Is the product of two elementary matrices an elementary matrix?

14. Let U and R be $n \times n$ upper triangular matrices and set $T = UR$. Show that T is also upper triangular and that $t_{jj} = u_{jj}r_{jj}$ for $j = 1, \ldots, n$.

15. Let A be a 3×3 matrix and suppose that

$$2\mathbf{a}_1 + \mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$$

How many solutions will the system $A\mathbf{x} = \mathbf{0}$ have? Explain. Is A nonsingular? Explain.

16. Let A be a 3×3 matrix and suppose that

$$\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3$$

Will the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Is A nonsingular? Explain your answers.

17. Let A and B be $n \times n$ matrices and let $C = A - B$. Show that if $A\mathbf{x}_0 = B\mathbf{x}_0$ and $\mathbf{x}_0 \neq \mathbf{0}$, then C must be singular.

18. Let A and B be $n \times n$ matrices and let $C = AB$. Prove that if B is singular, then C must be singular. [*Hint*: Use Theorem 5.2.]

19. Let U be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

(a) Explain why U must be nonsingular.

(b) Explain why U^{-1} must be upper triangular.

20. Let A be a nonsingular $n \times n$ matrix and let B be an $n \times r$ matrix. Show that the reduced row echelon form of $(A|B)$ is $(I|C)$, where $C = A^{-1}B$.

21. In general, matrix multiplication is not commutative (i.e., $AB \neq BA$). However, in certain special cases the commutative property does hold. Show that

(a) if D_1 and D_2 are $n \times n$ diagonal matrices, then $D_1D_2 = D_2D_1$.

(b) if A is an $n \times n$ matrix and

$$B = a_0I + a_1A + a_2A^2 + \cdots + a_kA^k$$

where $a_0, a_1, \ldots, a_k$ are scalars, then $AB = BA$.

22. Show that if A is a symmetric nonsingular matrix, then A^{-1} is also symmetric.

23. Prove that if A is row equivalent to B, then B is row equivalent to A.

24. **(a)** Prove that if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

(b) Prove that any two nonsingular $n \times n$ matrices are row equivalent.

25. Let A and B be $m \times n$ matrices. Prove that if B is row equivalent to A and U is any row echelon form A, then B is row equivalent to U.

26. Prove that B is row equivalent to A if and only if there exists a nonsingular matrix M such that $B = MA$.

27. Is it possible for a singular matrix B to be row equivalent to a nonsingular matrix A? Explain.

28. Given a vector $\mathbf{x} \in \mathbb{R}^{n+1}$, the $(n + 1) \times (n + 1)$ matrix V defined by

$$v_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ x_i^{j-1} & \text{for } j = 2, \ldots, n+1 \end{cases}$$

is called the Vandermonde matrix.

(a) Show that if

$$V\mathbf{c} = \mathbf{y}$$

and

$$p(x) = c_1 + c_2x + \cdots + c_{n+1}x^n$$

then

$$p(x_i) = y_i, \qquad i = 1, 2, \ldots, n+1$$

(b) Suppose that $x_1, x_2, \ldots, x_{n+1}$ are all distinct. Show that if $\mathbf{c}$ is a solution to $V\mathbf{x} = \mathbf{0}$, then the coefficients $c_1, c_2, \ldots, c_n$ must all be zero and hence V must be nonsingular.

For each of the following, answer true *if the statement is always true and answer* false *otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true.*

29. If A is row equivalent to I and $AB = AC$, then B must equal C.

30. If E and F are elementary matrices and $G = EF$, then G is nonsingular.

31. If A is a 4×4 matrix and $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3 + 2\mathbf{a}_4$, then A must be singular.

32. If A is row equivalent to both B and C, then A is row equivalent to $B + C$.

6 Partitioned Matrices

Often it is useful to think of a matrix as being composed of a number of submatrices. A matrix C can be partitioned into smaller matrices by drawing horizontal lines between the rows and vertical lines between the columns. The smaller matrices are often referred to as *blocks*. For example, let

$$C = \begin{pmatrix} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{pmatrix}$$

If lines are drawn between the second and third rows and between the third and fourth columns, then C will be divided into four submatrices, C_{11}, C_{12}, C_{21}, and C_{22}:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \left(\begin{array}{ccc|cc} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{array}\right)$$

One useful way of partitioning a matrix is into columns. For example, if

$$B = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$

then we can partition B into three column submatrices:

$$B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = \left(\begin{array}{c|c|c} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{array}\right)$$

Suppose that we are given a matrix A with three columns; then the product AB can be viewed as a block multiplication. Each block of B is multiplied by A, and the result is a matrix with three blocks: $A\mathbf{b}_1$, $A\mathbf{b}_2$, and $A\mathbf{b}_3$; that is,

$$AB = A(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (A\mathbf{b}_1, A\mathbf{b}_2, A\mathbf{b}_3)$$

For example, if

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$