Using the trapezoidal rule formula (7), we may also approximate the line integral by

$$
\begin{aligned}
T_4 &= \left(0 + \tfrac{1}{2}\sqrt{1+\left(\tfrac{1}{2}\right)^3}\right)\frac{1/2}{2} + \left(\tfrac{1}{2}\sqrt{1+\left(\tfrac{1}{2}\right)^3} + 1\sqrt{1+1^3}\right)\frac{1/2}{2} \\
&\quad + \left(1\sqrt{1+1^3} + \tfrac{3}{2}\sqrt{1+\left(\tfrac{3}{2}\right)^3}\right)\frac{1/2}{2} + \left(\tfrac{3}{2}\sqrt{1+\left(\tfrac{3}{2}\right)^3} + 2\sqrt{1+2^3}\right)\frac{1/2}{2} \\
&\quad + (e^0 + e^{-1/4})\frac{1/2}{2} + (e^{-1/4} + e^{-1})\frac{1/2}{2} + (e^{-1} + e^{-9/4})\frac{1/2}{2} \\
&\quad + (e^{-9/4} + e^{-4})\frac{1/2}{2} + (\cos 0 + \cos\tfrac{1}{4})\frac{1/2}{2} + (\cos\tfrac{1}{4} + \cos 1)\frac{1/2}{2} \\
&\quad + (\cos 1 + \cos\tfrac{9}{4})\frac{1/2}{2} + (\cos\tfrac{9}{4} + \cos 4)\frac{1/2}{2} \\
&\approx 5.44874.
\end{aligned}
$$

As was the case in Example 12, because of the small number of sampling points used, formulas (6) and (7) provide relatively crude approximations. ◆

1 Exercises

1. Let $f(x, y) = x + 2y$. Evaluate the scalar line integral $\int_{\mathbf{x}} f\, ds$ over the given path $\mathbf{x}$.

(a) $\mathbf{x}(t) = (2 - 3t, 4t - 1), 0 \le t \le 2$

(b) $\mathbf{x}(t) = (\cos t, \sin t), 0 \le t \le \pi$

In Exercises 2–7, calculate $\int_{\mathbf{x}} f\, ds$, where f and $\mathbf{x}$ are as indicated.

2. $f(x, y, z) = xyz, \mathbf{x}(t) = (t, 2t, 3t), 0 \le t \le 2$

3. $f(x, y, z) = \dfrac{x+z}{y+z}, \mathbf{x}(t) = (t, t, t^{3/2}), 1 \le t \le 3$

4. $f(x, y, z) = 3x + xy + z^3, \mathbf{x}(t) = (\cos 4t, \sin 4t, 3t)$, $0 \le t \le 2\pi$

5. $f(x, y, z) = \dfrac{z}{x^2 + y^2}, \mathbf{x}(t) = (e^{2t}\cos 3t, e^{2t}\sin 3t, e^{2t})$, $0 \le t \le 5$

6. $f(x, y, z) = x + y + z$,

$$\mathbf{x}(t) = \begin{cases} (2t, 0, 0) & \text{if } 0 \le t \le 1 \\ (2, 3t - 3, 0) & \text{if } 1 \le t \le 2 \\ (2, 3, 2t - 4) & \text{if } 2 \le t \le 3 \end{cases}$$

7. $f(x, y, z) = 2x - y^{1/2} + 2z^2$,

$$\mathbf{x}(t) = \begin{cases} (t, t^2, 0) & \text{if } 0 \le t \le 1 \\ (1, 1, t - 1) & \text{if } 1 \le t \le 3 \end{cases}$$

In Exercises 8–16, find $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$, where the vector field $\mathbf{F}$ and the path $\mathbf{x}$ are given.

8. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{x}(t) = (2t + 1, t, 3t - 1)$, $0 \le t \le 1$

9. $\mathbf{F} = (y + 2)\,\mathbf{i} + x\,\mathbf{j}$, $\mathbf{x}(t) = (\sin t, -\cos t)$, $0 \le t \le \pi/2$

10. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}, \mathbf{x}(t) = (2t + 1, t + 2), 0 \le t \le 1$

11. $\mathbf{F} = (y - x)\,\mathbf{i} + x^4y^3\,\mathbf{j}, \mathbf{x}(t) = (t^2, t^3), -1 \le t \le 1$

12. $\mathbf{F} = x\,\mathbf{i} + xy\,\mathbf{j} + xyz\,\mathbf{k}$, $\mathbf{x}(t) = (3\cos t, 2\sin t, 5t)$, $0 \le t \le 2\pi$

13. $\mathbf{F} = -3y\,\mathbf{i} + x\,\mathbf{j} + 3z^2\,\mathbf{k}$, $\mathbf{x}(t) = (2t + 1, t^2 + t, e^t)$, $0 \le t \le 1$

14. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - z\,\mathbf{k}, \mathbf{x}(t) = (t, 3t^2, 2t^3), -1 \le t \le 1$

15. $\mathbf{F} = 3z\,\mathbf{i} + y^2\,\mathbf{j} + 6z\,\mathbf{k}$, $\mathbf{x}(t) = (\cos t, \sin t, t/3)$, $0 \le t \le 4\pi$

16. $\mathbf{F} = y\cos z\,\mathbf{i} + x\sin z\,\mathbf{j} + xy\sin z^2\,\mathbf{k}$, $\mathbf{x}(t) = (t, t^2, t^3), 0 \le t \le 1$

17. Determine the value of $\int_{\mathbf{x}} x\,dy - y\,dx$, where $\mathbf{x}(t) = (\cos 3t, \sin 3t), 0 \le t \le \pi$.

18. Find the work done by the force field $\mathbf{F} = 2x\mathbf{i} + \mathbf{j}$ when a particle moves along the path $\mathbf{x}(t) = (t, 3t^2, 2), 0 \le t \le 2$.

19. If $\mathbf{x} = (e^{2t}\cos 3t, e^{2t}\sin 3t), \quad 0 \le t \le 2\pi,$ find $\displaystyle\int_{\mathbf{x}} \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}}$.

20. Let C be the portion of the curve $y = 2\sqrt{x}$ between $(1, 2)$ and $(9, 6)$. Find $\int_C 3y\,ds$.

21. Let $\mathbf{F} = (x^2 + y)\mathbf{i} + (y - x)\mathbf{j}$ and consider the two paths

$$\mathbf{x}(t) = (t, t^2),\ 0 \le t \le 1 \quad \text{and}$$
$$\mathbf{y}(t) = (1 - 2t, 4t^2 - 4t + 1),\ 0 \le t \le \tfrac{1}{2}.$$

(a) Calculate $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ and $\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$.

(b) By considering the image curves of the paths $\mathbf{x}$ and $\mathbf{y}$, discuss your answers in part (a).

22. Find the work done by the force field $\mathbf{F} = x^2 y\,\mathbf{i} + z\,\mathbf{j} + (2x - y)\,\mathbf{k}$ on a particle as the particle moves along a straight line from $(1, 1, 1)$ to $(2, -3, 3)$.

23. Let $\mathbf{F} = (2z^5 - 3xy)\,\mathbf{i} - x^2\,\mathbf{j} + x^2 z\,\mathbf{k}$. Calculate the line integral of $\mathbf{F}$ around the perimeter of the square with vertices $(1, 1, 3)$, $(-1, 1, 3)$, $(-1, -1, 3)$, $(1, -1, 3)$, oriented counterclockwise about the z-axis.

24. Evaluate $\int_C (x^2 - y)\,dx + (x - y^2)\,dy$, where C is the line segment from $(1, 1)$ to $(3, 5)$.

25. Find $\int_C x^2 y\,dx - (x + y)\,dy$, where C is the trapezoid with vertices $(0, 0)$, $(3, 0)$, $(3, 1)$, and $(1, 1)$, oriented counterclockwise.

26. Evaluate $\int_C x^2 y\,dx - xy\,dy$, where C is the curve with equation $y^2 = x^3$, from $(1, -1)$ to $(1, 1)$.

27. Evaluate $\int_C y\,dx - x\,dy$, where C is the portion of the parabola $y = x^2$, from $(3, 9)$ to $(0, 0)$.

28. Evaluate $\int_C (x - y)^2\,dx + (x + y)^2\,dy$, where C is the portion of $y = |x|$, from $(-2, 2)$ to $(1, 1)$.

29. Evaluate $\int_C xy^2\,dx - xy\,dy$, where C is the semicircular arc from $(0, 2)$ to $(0, -2)$ traveled clockwise.

30. Find the circulation of $\mathbf{F} = (x^2 - y)\,\mathbf{i} + (xy + x)\,\mathbf{j}$ along the circle $x^2 + y^2 = 16$, oriented counterclockwise.

31. Evaluate $\int_C yz\,dx - xz\,dy + xy\,dz$, where C is the line segment from $(1, 1, 2)$ to $(5, 3, 1)$.

32. Calculate $\int_C z\,dx + x\,dy + y\,dz$, where C is the curve obtained by intersecting the surfaces $z = x^2$ and $x^2 + y^2 = 4$ and oriented counterclockwise around the z-axis (as seen from the positive z-axis).

33. Show that $\int_{\mathbf{x}} \mathbf{T} \cdot d\mathbf{s}$ equals the length of the path $\mathbf{x}$, where $\mathbf{T}$ denotes the unit tangent vector of the path.

34. Tom Sawyer is whitewashing a picket fence. The bases of the fenceposts are arranged in the xy-plane as the quarter circle $x^2 + y^2 = 25$, $x, y \ge 0$, and the height of the fencepost at point (x, y) is given by $h(x, y) = 10 - x - y$ (units are feet). Use a scalar line integral to find the area of one side of the fence. (See Figure 16.)

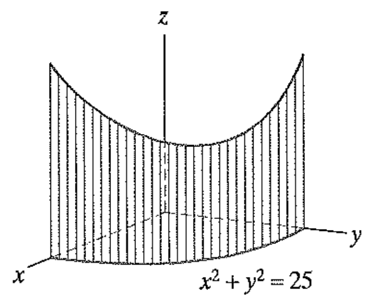

Figure 16 The picket fence of Exercise 34. The base of the fence is the quarter circle $x^2 + y^2 = 25, x, y \ge 0$.

35. Sisyphus is pushing a boulder up a 100-ft-tall spiral staircase surrounding a cylindrical castle tower. (See Figure 17.)

(a) Suppose Sisyphus's path is described parametrically as

$$\mathbf{x}(t) = (5\cos 3t, 5\sin 3t, 10t), \quad 0 \le t \le 10.$$

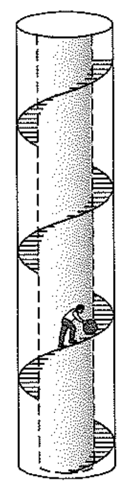

Figure 17 Sisyphus's path up the spiral staircase of Exercise 35.

If he exerts a force with a constant magnitude of 50 lb tangent to his path, find the work Sisyphus