Step 3. Unfortunately, not all regions described in the statement of Theorem 2.1 can be subdivided into finitely many elementary regions of type 3. Here is an outline of what we might do to prove Green's theorem in such generality.

First, we claim (without proof) that for regions D described in the statement of Theorem 2.1 we can produce a sequence of regions $D_1, D_2, \ldots, D_n, \ldots$ whose "limit" as $n \to \infty$ is D and such that each D_n can be subdivided into finitely many type 3 elementary regions. Next, we claim that $\partial D_n \to \partial D$ as $n \to \infty$. Finally, we need to prove that, as $n \to \infty$,

$$\iint_{D_n} (N_x - M_y)\, dA \quad \to \quad \iint_D (N_x - M_y)\, dA$$

and

$$\oint_{\partial D_n} M\, dx + N\, dy \quad \to \quad \oint_{\partial D} M\, dx + N\, dy.$$

Since Green's theorem holds for each D_n (by Steps 1 and 2), we are done.[2] ■

Historical Note[3]

The idea that the line integral of a vector field along a closed curve can be related to a double integral over the region bounded by the curve is frequently attributed to George Green (1793–1841), a self-educated English mathematician. The result we have been calling Green's theorem had its origins in a rather obscure 1828 pamphlet published by Green, in which he sought to lay a rigorous mathematical foundation for the physics of electricity and magnetism. Green's ideas arose from work in partial differential equations concerning gravitational potentials. Green's pamphlet subsequently came to the attention of Lord Kelvin (1824–1907), who had it republished so that, fortunately, Green's results received greater recognition.

Coincidentally, a result similar to Green's theorem was established independently (and also in 1828!) by the Russian mathematician Mikhail Ostrogradsky (1801–1861). Ostrogradsky's name is sometimes associated to what we call Green's theorem.

2 Exercises

In Exercises 1–6, verify Green's theorem for the given vector field

$$\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$$

and region D by calculating both

$$\oint_{\partial D} M\, dx + N\, dy \quad \text{and} \quad \iint_D (N_x - M_y)\, dA.$$

1. $\mathbf{F} = -x^2 y\,\mathbf{i} + xy^2\,\mathbf{j}$, D is the disk $x^2 + y^2 \le 4$.
2. $\mathbf{F} = (x^2 - y)\,\mathbf{i} + (x + y^2)\,\mathbf{j}$, D is the rectangle bounded by $x = 0$, $x = 2$, $y = 0$, and $y = 1$.
3. $\mathbf{F} = y\,\mathbf{i} + x^2\,\mathbf{j}$, D is the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$.
4. $\mathbf{F} = 2y\,\mathbf{i} + x\,\mathbf{j}$, D is the semicircular region $x^2 + y^2 \le a^2$, $y \ge 0$.
5. $\mathbf{F} = 3y\,\mathbf{i} - 4x\,\mathbf{j}$, D is the elliptical region $x^2 + 2y^2 \le 4$.

[2] For details of the type of limit argument we have in mind, see O. D. Kellogg, *Foundations of Potential Theory* (Springer, Berlin, 1929; reprinted by Dover Publications, New York, 1954), pp. 113–119, where a limit argument is given in the case of Gauss's theorem. For a proof of Green's theorem that avoids the limit argument, see D. V. Widder, *Advanced Calculus*, 2nd ed., (Prentice-Hall, Englewood Cliffs, 1961; reprinted by Dover Publications, New York, 1989), pp. 223–225.

[3] See also M. Kline, *Mathematical Thought from Ancient to Modern Times* (Oxford Press, New York, 1972), p. 683.

6. $\mathbf{F} = (x^2y + x)\,\mathbf{i} + (y^3 - xy^2)\,\mathbf{j}$, D is the region inside the circle $x^2 + y^2 = 9$ and outside the circle $x^2 + y^2 = 4$.

7. (a) Use Green's theorem to calculate the line integral

$$\oint_C y^2 dx + x^2 dy,$$

where C is the path formed by the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, oriented counterclockwise.

(b) Verify your answer for part (a) by calculating the line integral directly.

8. Let $\mathbf{F} = 3xy\,\mathbf{i} + 2x^2\,\mathbf{j}$ and suppose C is the oriented curve shown in Figure 28. Evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{s}$$

both directly and also by means of Green's theorem.

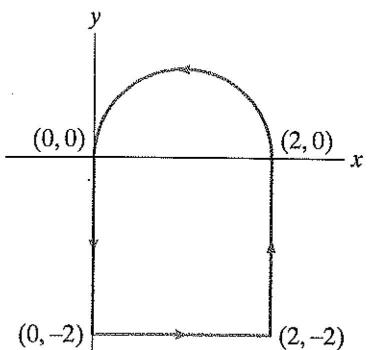

Figure 28 The oriented curve C of Exercise 8 consists of three sides of a square plus a semicircular arc.

9. Evaluate

$$\oint_C (x^2 - y^2)\,dx + (x^2 + y^2)\,dy,$$

where C is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, oriented *clockwise*. Use whatever method of evaluation seems appropriate.

Use Green's theorem to find the work done by the vector field

$$\mathbf{F} = (4y - 3x)\,\mathbf{i} + (x - 4y)\,\mathbf{j}$$

on a particle as the particle moves counterclockwise once around the ellipse $x^2 + 4y^2 = 4$.

Verify that the area of the rectangle $R = [0, a] \times [0, b]$ is ab, by calculating an appropriate line integral.

12. Let a be a positive constant. Use Green's theorem to calculate the area under one arch of the cycloid

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

13. Evaluate $\oint_C (x^4y^5 - 2y)\,dx + (3x + x^5y^4)\,dy$, where C is the oriented curve pictured in Figure 29.

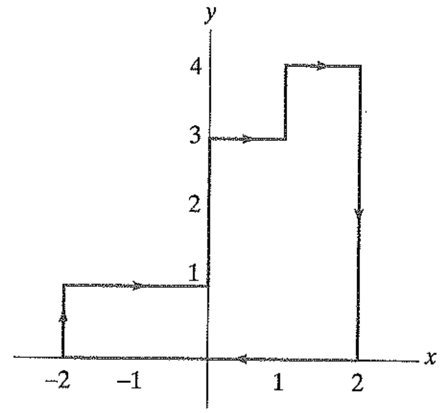

Figure 29 The oriented curve C of Exercise 13.

14. Use Green's theorem to find the area enclosed by the hypocycloid

$$\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t), \quad 0 \le t \le 2\pi.$$

15. (a) Sketch the curve given parametrically by $\mathbf{x}(t) = (1 - t^2, t^3 - t)$.

(b) Find the area inside the closed loop of the curve.

16. Use Green's theorem to find the area between the ellipse $x^2/9 + y^2/4 = 1$ and the circle $x^2 + y^2 = 25$.

17. Show that if D is a region to which Green's theorem applies, and ∂D is oriented so that D is always on the left as we travel along ∂D, then the area of D is given by either of the following two line integrals:

$$\text{Area of } D = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx.$$

18. Find the area inside the quadrilateral whose vertices taken counterclockwise are $(2, 0)$, $(1, 2)$, $(-1, 1)$, and $(1, 1)$.

19. Suppose that the successive vertices of an n-sided polygon are the points (a_1, b_1), (a_2, b_2), $\ldots (a_n, b_n)$, arranged counterclockwise around the polygon. Show that the area inside the polygon is given by

$$\frac{1}{2}\left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + \cdots + \begin{vmatrix} a_{n-1} & b_{n-1} \\ a_n & b_n \end{vmatrix} + \begin{vmatrix} a_n & b_n \\ a_1 & b_1 \end{vmatrix}\right).$$

20. Let a be a positive integer throughout this problem. An epicycloid is the path produced by a marked point on a circle of unit radius that rolls, without slipping, on the outside of a fixed circle of radius a. If the center of the fixed circle is at the origin and the marked point is at $(a, 0)$ when $t = 0$, then the epicycloid is given by the path $\mathbf{x}(t) = ((a+1)\cos t - \cos(a+1)t, (a+1)\sin t - \sin(a+1)t)$.

(a) Show that the epicycloid path meets the fixed circle exactly when $t = 2\pi n/a$, where n is an integer. (Hint: This must happen when $\|\mathbf{x}(t)\| = a$.) Graph the epicycloid when $a = 5, 6$.

(b) Use an appropriate line integral to find the area enclosed by the epicycloid.

(c) As the integer a gets larger, what happens to the ratio of the area calculated in part (b) to that of the fixed circle?

21. Evaluate the line integral $\oint_C 5y\,dx - 3x\,dy$, where C is the cardioid with polar coordinate equation $r = 1 - \sin\theta$, oriented counterclockwise.

22. (a) Suppose that C is a simple, closed curve that does *not* enclose the origin. Use Green's theorem to determine the value of

$$\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2}.$$

(b) Now suppose that C is a simple, closed curve that *does* enclose the origin. Can you use Green's theorem to determine the value of

$$\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2}?$$

Explain.

(c) Let C_1 and C_2 be two simple, closed curves that both enclose the origin, are both oriented counterclockwise, and do not touch or intersect. Show that

$$\oint_{C_1} \frac{x\,dx + y\,dy}{x^2 + y^2} = \oint_{C_2} \frac{x\,dx + y\,dy}{x^2 + y^2}.$$

(d) Use the result of part (c) to determine the value of

$$\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2},$$

where C is a simple, closed curve that encloses the origin.

23. (a) Use the divergence theorem (Theorem 2.3) to show that $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$, where $\mathbf{F} = 2y\,\mathbf{i} - 3x\,\mathbf{j}$ and C is the circle $x^2 + y^2 = 1$.

(b) Now show $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$ by direct computation of the line integral.

24. Let $\mathbf{F} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$. The divergence theorem shows that the flux of $\mathbf{F}$ across a closed curve C (i.e., $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$) is equal to $\iint_D (M_x + N_y)\,dA$, where D is the region bounded by C. Use Green's theorem to establish a similar result involving $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$, the circulation of $\mathbf{F}$ along C. (See also §1.)

25. Let C be any simple, closed curve in the plane. Show that

$$\oint_C 3x^2y\,dx + x^3\,dy = 0.$$

26. Show that

$$\oint_C -y^3\,dx + (x^3 + 2x + y)\,dy$$

is positive for any closed curve C to which Green's theorem applies.

27. Show that if C is the boundary of any rectangular region in $\mathbf{R}^2$, then

$$\oint_C (x^2y^3 - 3y)\,dx + x^3y^2\,dy$$

depends only on the area of the rectangle, not on its placement in $\mathbf{R}^2$.

28. Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ be the position vector of any point in the plane. Show that the flux of $\mathbf{F} = \mathbf{r}$ across any simple closed curve C in $\mathbf{R}^2$ is twice the area inside C.

29. Let D be a region to which Green's theorem applies and suppose that $u(x, y)$ and $v(x, y)$ are two functions of class C^2 whose domains include D. Show that

$$\iint_D \frac{\partial(u, v)}{\partial(x, y)}\,dA = \oint_C (u\nabla v)\cdot d\mathbf{s},$$

where $C = \partial D$ is oriented as in Green's theorem.

30. Let $f(x, y)$ be a function of class C^2 such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

(i.e., f is *harmonic*). Show that if C is any closed curve to which Green's theorem applies, then

$$\oint_C \frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy = 0.$$

31. Let D be a region to which Green's theorem applies and $\mathbf{n}$ the outward unit normal vector to D. Suppose $f(x, y)$ is a function of class C^2. Show that

$$\iint_D \nabla^2 f\,dA = \oint_{\partial D} \frac{\partial f}{\partial n}\,ds,$$

where $\nabla^2 f$ denotes the Laplacian of f (namely, $\nabla^2 f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$) and $\partial f/\partial n$ denotes $\nabla f\cdot\mathbf{n}$. (See the proof of Theorem 2.3 for more information about $\mathbf{n}$.)