Proof The proof is by induction on n. Clearly, the result holds if $n = 1$, since a 1×1 matrix is necessarily symmetric. Assume that the result holds for all $k \times k$ matrices and that A is a $(k+1) \times (k+1)$ matrix. Expanding $\det(A)$ along the first row of A, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1})$$

Since the M_{ij}'s are all $k \times k$ matrices, it follows from the induction hypothesis that

$$\det(A) = a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1}^T) \qquad (9)$$

The right-hand side of (9) is just the expansion by minors of $\det(A^T)$ using the first column of A^T. Therefore,

$$\det(A^T) = \det(A)$$

■

Theorem 1.3 If A is an $n \times n$ triangular matrix, then the determinant of A equals the product of the diagonal elements of A.

Proof In view of Theorem 1.2, it suffices to prove the theorem for lower triangular matrices. The result follows easily using the cofactor expansion and induction on n. The details are left for the reader (see Exercise 8 at the end of the section). ■

Theorem 1.4 Let A be an $n \times n$ matrix.

(i) If A has a row or column consisting entirely of zeros, then $\det(A) = 0$.

(ii) If A has two identical rows or two identical columns, then $\det(A) = 0$.

Both of these results can be easily proved with the use of the cofactor expansion. The proofs are left for the reader (see Exercises 9 and 10 at the end of the section). ■

In the next section, we look at the effect of row operations on the value of the determinant. This will allow us to make use of Theorem 1.3 to derive a more efficient method for computing the value of a determinant.

SECTION 1 EXERCISES

1. Let

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

(a) Find the values of $\det(M_{21})$, $\det(M_{22})$, and $\det(M_{23})$.

(b) Find the values of A_{21}, A_{22}, and A_{23}.

(c) Use your answers from part (b) to compute $\det(A)$.

2. Use determinants to determine whether the following 2×2 matrices are nonsingular:

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -6 \\ 2 & 4 \end{bmatrix}$

3. Evaluate the following determinants:

(a) $\begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix}$ **(b)** $\begin{vmatrix} 5 & -2 \\ -8 & 4 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{vmatrix}$ **(d)** $\begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{vmatrix}$ (f) $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{vmatrix}$

(g) $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

(h) $\begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{vmatrix}$

4. Evaluate the following determinants by inspection:

(a) $\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 3 & -2 \end{vmatrix}$

(c) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 4 & 0 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 3 \end{vmatrix}$

5. Evaluate the following determinant. Write your answer as a polynomial in x.

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

6. Find all values of λ for which the following determinant will equal 0:

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

7. Let A be a 3×3 matrix with $a_{11} = 0$ and $a_{21} \neq 0$. Show that A is row equivalent to I if and only if

$$\begin{aligned} &-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} \\ &+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0 \end{aligned}$$

8. Write out the details of the proof of Theorem 1.3.

9. Prove that if a row or a column of an $n \times n$ matrix A consists entirely of zeros, then $\det(A) = 0$.

10. Use mathematical induction to prove that if A is an $(n+1) \times (n+1)$ matrix with two identical rows, then $\det(A) = 0$.

11. Let A and B be 2×2 matrices.

(a) Does $\det(A + B) = \det(A) + \det(B)$?

(b) Does $\det(AB) = \det(A)\det(B)$?

(c) Does $\det(AB) = \det(BA)$?

Justify your answers.

12. Let A and B be 2×2 matrices and let

$$C = \begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad D = \begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

(a) Show that $\det(A + B) = \det(A) + \det(B) + \det(C) + \det(D)$.

(b) Show that if $B = EA$ then $\det(A + B) = \det(A) + \det(B)$.

13. Let A be a symmetric tridiagonal matrix (i.e., A is symmetric and $a_{ij} = 0$ whenever $|i - j| > 1$). Let B be the matrix formed from A by deleting the first two rows and columns. Show that

$$\det(A) = a_{11}\det(M_{11}) - a_{12}^2\det(B)$$

2 Properties of Determinants

In this section, we consider the effects of row operations on the determinant of a matrix. Once these effects have been established, we will prove that a matrix A is singular if and only if its determinant is zero, and we will develop a method for evaluating determinants by using row operations. Also, we will establish an important theorem about the determinant of the product of two matrices. We begin with the following lemma:

Lemma 2.1 Let A be an $n \times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, \ldots, n$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (1)$$