SECTION 2 EXERCISES

1. Evaluate each of the following determinants by inspection:

(a) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

2. Let

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix}$$

(a) Use the elimination method to evaluate $\det(A)$.

(b) Use the value of $\det(A)$ to evaluate

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix}$$

3. For each of the following, compute the determinant and state whether the matrix is singular or nonsingular:

(a) $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix}$

4. Find all possible choices of c that would make the following matrix singular:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}$$

5. Let A be an $n \times n$ matrix and α a scalar. Show that

$$\det(\alpha A) = \alpha^n \det(A)$$

6. Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

7. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of

(a) $\det(AB)$ **(b)** $\det(3A)$

(c) $\det(2AB)$ **(d)** $\det(A^{-1}B)$

8. Show that if E is an elementary matrix, then E^T is an elementary matrix of the same type as E.

9. Let E_1, E_2, and E_3 be 3×3 elementary matrices of types I, II, and III, respectively, and let A be a 3×3 matrix with $\det(A) = 6$. Assume, additionally, that E_2 was formed from I by multiplying its second row by 3. Find the values of each of the following:

(a) $\det(E_1 A)$ **(b)** $\det(E_2 A)$

(c) $\det(E_3 A)$ **(d)** $\det(AE_1)$

(e) $\det(E_1^2)$ **(f)** $\det(E_1 E_2 E_3)$

10. Let A and B be row equivalent matrices, and suppose that B can be obtained from A by using only row operations I and III. How do the values of $\det(A)$ and $\det(B)$ compare? How will the values compare if B can be obtained from A by using only row operation III? Explain your answers.

11. Let A be an $n \times n$ matrix. Is it possible for $A^2 + I = O$ in the case where n is odd? Answer the same question in the case where n is even.

12. Consider the 3×3 Vandermonde matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

(a) Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$. [*Hint*: Make use of row operation III.]

(b) What conditions must the scalars x_1, x_2, and x_3 satisfy in order for V to be nonsingular?

13. Suppose that a 3×3 matrix A factors into a product

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Determine the value of $\det(A)$.

14. Let A and B be $n \times n$ matrices. Prove that the product AB is nonsingular if and only if A and B are both nonsingular.

15. Let A and B be $n \times n$ matrices. Prove that if $AB = I$, then $BA = I$. What is the significance of this result in terms of the definition of a nonsingular matrix?

16. A matrix A is said to be *skew symmetric* if $A^T = -A$. For example,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is skew symmetric, since

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

If A is an $n \times n$ skew-symmetric matrix and n is odd, show that A must be singular.

17. Let A be a nonsingular $n \times n$ matrix with a nonzero cofactor A_{nn}, and set

$$c = \frac{\det(A)}{A_{nn}}$$

Show that if we subtract c from a_{nn}, then the resulting matrix will be singular.

18. Let A be a $k \times k$ matrix and let B be an $(n-k) \times (n-k)$ matrix. Let

$$E = \begin{bmatrix} I_k & O \\ O & B \end{bmatrix}, \quad F = \begin{bmatrix} A & O \\ O & I_{n-k} \end{bmatrix},$$
$$C = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$$

where I_k and I_{n-k} are the $k \times k$ and $(n-k) \times (n-k)$ identity matrices.

(a) Show that $\det(E) = \det(B)$.

(b) Show that $\det(F) = \det(A)$.

(c) Show that $\det(C) = \det(A)\det(B)$.

19. Let A and B be $k \times k$ matrices and let

$$M = \begin{bmatrix} O & B \\ A & O \end{bmatrix}$$

Show that $\det(M) = (-1)^k \det(A)\det(B)$.

20. Show that evaluating the determinant of an $n \times n$ matrix by cofactors involves $(n! - 1)$ additions and $\sum_{k=1}^{n-1} n!/k!$ multiplications.

21. Show that the elimination method of computing the value of the determinant of an $n \times n$ matrix involves $[n(n-1)(2n-1)]/6$ additions and $[(n-1)(n^2+n+3)]/3$ multiplications and divisions. [*Hint*: At the ith step of the reduction process, it takes $n - i$ divisions to calculate the multiples of the ith row that are to be subtracted from the remaining rows below the pivot. We must then calculate new values for the $(n-i)^2$ entries in rows $i+1$ through n and columns $i+1$ through n.]

3 Additional Topics and Applications

In this section, we learn a method for computing the inverse of a nonsingular matrix A using determinants and we learn a method for solving linear systems using determinants. Both methods depend on Lemma 2.1. We also show how to use determinants to define the cross product of two vectors. The cross product is useful in physics applications involving the motion of a particle in 3-space.