from (4) that

$$\|\mathbf{B}(t)\| = \|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\|\|\mathbf{N}(t)\| \sin\frac{\pi}{2} = 1$$

The vector $\mathbf{B}(t)$ defined by (5) is called the *binormal* vector (see Figure 3.3).

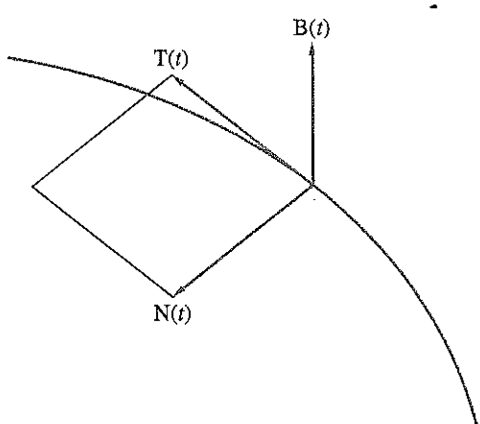

Figure 3.3.

SECTION 3 EXERCISES

1. For each of the following, compute (i) $\det(A)$, (ii) adj A, and (iii) A^{-1}:

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

2. Use Cramer's rule to solve each of the following systems:

(a) $\begin{aligned} x_1 + 2x_2 &= 3 \\ 3x_1 - x_2 &= 1 \end{aligned}$ **(b)** $\begin{aligned} 2x_1 + 3x_2 &= 2 \\ 3x_1 + 2x_2 &= 5 \end{aligned}$

(c) $\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ 4x_1 + 5x_2 + x_3 &= 8 \\ -2x_1 - x_2 + 4x_3 &= 2 \end{aligned}$

(d) $\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 2x_2 - x_3 &= -8 \end{aligned}$

(e) $\begin{aligned} x_1 + x_2 \qquad\qquad &= 0 \\ x_2 + x_3 - 2x_4 &= 1 \\ x_1 \qquad + 2x_3 + x_4 &= 0 \\ x_1 + x_2 \qquad + x_4 &= 0 \end{aligned}$

3. Given

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$$

determine the (2, 3) entry of A^{-1} by computing a quotient of two determinants.

4. Let A be the matrix in Exercise 3. Compute the third column of A^{-1} by using Cramer's rule to solve $A\mathbf{x} = \mathbf{e}_3$.

5. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

(a) Compute the determinant of A. Is A nonsingular?

(b) Compute adj A and the product A adj A.

6. If A is singular, what can you say about the product A adj A?

7. Let B_j denote the matrix obtained by replacing the jth column of the identity matrix with a vector $\mathbf{b} = (b_1, \ldots, b_n)^T$. Use Cramer's rule to show that

$$b_j = \det(B_j) \quad \text{for } j = 1, \ldots, n$$

8. Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that

$$\det(\text{adj } A) = (\det(A))^{n-1}$$

9. Let A be a 4×4 matrix. If

$$\text{adj } A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{bmatrix}$$

(a) calculate the value of det(adj A). What should the value of $\det(A)$ be? [*Hint*: Use the result from Exercise 8.]

(b) find A.

10. Show that if A is nonsingular, then adj A is nonsingular and

$$(\text{adj } A)^{-1} = \det(A^{-1})A = \text{adj } A^{-1}$$

11. Show that if A is singular, then adj A is also singular.

12. Show that if $\det(A) = 1$, then

$$\text{adj}(\text{adj } A) = A$$

13. Suppose that Q is a matrix with the property $Q^{-1} = Q^T$. Show that

$$q_{ij} = \frac{Q_{ij}}{\det(Q)}$$

14. In coding a message, a blank space was represented by 0, an A by 1, a B by 2, a C by 3, and so on. The message was transformed using the matrix

$$A = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

and sent as

$$-19, 19, 25, -21, 0, 18, -18, 15, 3, 10,$$
$$-8, 3, -2, 20, -7, 12$$

What was the message?

15. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Show each of the following:

(a) $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ (b) $\mathbf{y} \times \mathbf{x} = -(\mathbf{x} \times \mathbf{y})$

(c) $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z})$

(d) $\mathbf{z}^T(\mathbf{x} \times \mathbf{y}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

16. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$ and define the skew-symmetric matrix A_x by

$$A_x = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

(a) Show that $\mathbf{x} \times \mathbf{y} = A_x\mathbf{y}$.

(b) Show that $\mathbf{y} \times \mathbf{x} = A_x^T\mathbf{y}$.

Chapter Exercises

MATLAB EXERCISES

The first four exercises that follow involve integer matrices and illustrate some of the properties of determinants that were covered in this chapter. The last two exercises illustrate some of the differences that may arise when we work with determinants in floating-point arithmetic.

In theory, the value of the determinant should tell us whether the matrix is nonsingular. However, if the matrix is singular and its determinant is computed using finite-precision arithmetic, then, because of roundoff errors, the computed value of the determinant may not equal zero. A computed value near zero does not necessarily mean that the matrix is singular or even close to being singular. Furthermore, a matrix may be nearly singular and have a determinant that is not even close to zero (see Exercise 6).

1. Generate random 5×5 matrices with integer entries by setting

$$A = \texttt{round}(10 * \texttt{rand}(5))$$

and

$$B = \texttt{round}(20 * \texttt{rand}(5)) - 10$$