SECTION 1 EXERCISES

1. Consider the vectors $\mathbf{x}_1 = (8,\ 6)^T$ and $\mathbf{x}_2 = (4,\ -1)^T$ in $\mathbb{R}^2$.
   (a) Determine the length of each vector.
   (b) Let $\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2$. Determine the length of $\mathbf{x}_3$. How does this length compare with the sum of the lengths of $\mathbf{x}_1$ and $\mathbf{x}_2$?
   (c) Draw a graph illustrating how $\mathbf{x}_3$ can be constructed geometrically using $\mathbf{x}_1$ and $\mathbf{x}_2$. Use this graph to give a geometrical interpretation of your answer to the question in part (b).
2. Repeat Exercise 1 for the vectors $\mathbf{x}_1 = (2,\ 1)^T$ and $\mathbf{x}_2 = (6,\ 3)^T$.
3. Let C be the set of complex numbers. Define addition on C by
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
and define scalar multiplication by
$$\alpha(a + bi) = \alpha a + \alpha bi$$
for all real numbers α. Show that C is a vector space with these operations.
4. Show that $\mathbb{R}^{m\times n}$, together with the usual addition and scalar multiplication of matrices, satisfies the eight axioms of a vector space.
5. Show that $C[a, b]$, together with the usual scalar multiplication and addition of functions, satisfies the eight axioms of a vector space.
6. Let P be the set of all polynomials. Show that P, together with the usual addition and scalar multiplication of functions, forms a vector space.
7. Show that the element $\mathbf{0}$ in a vector space is unique.
8. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in a vector space V. Prove that if
$$\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$$
then $\mathbf{y} = \mathbf{z}$.
9. Let V be a vector space and let $\mathbf{x} \in V$. Show that
   (a) $\beta\mathbf{0} = \mathbf{0}$ for each scalar β.
   (b) if $\alpha\mathbf{x} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{x} = \mathbf{0}$.
10. Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by
$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$
$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$$
We use the symbol $\oplus$ to denote the addition operation for this system in order to avoid confusion with the usual addition $\mathbf{x} + \mathbf{y}$ of row vectors. Show that S, together with the ordinary scalar multiplication and the addition operation $\oplus$, is not a vector space. Which of the eight axioms fail to hold?
11. Let V be the set of all ordered pairs of real numbers with addition defined by
$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
and scalar multiplication defined by
$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$$
Scalar multiplication for this system is defined in an unusual way, and consequently we use the symbol $\circ$ to avoid confusion with the ordinary scalar multiplication of row vectors. Is V a vector space with these operations? Justify your answer.
12. Let $\mathbb{R}^+$ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by
$$\alpha \circ x = x^\alpha$$
for each $x \in \mathbb{R}^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by
$$x \oplus y = x \cdot y \qquad \text{for all} \qquad x, y \in \mathbb{R}^+$$
Thus, for this system, the scalar product of -3 times $\frac{1}{2}$ is given by
$$-3 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-3} = 8$$
and the sum of 2 and 5 is given by
$$2 \oplus 5 = 2 \cdot 5 = 10$$
Is $\mathbb{R}^+$ a vector space with these operations? Prove your answer.
13. Let $\mathbb{R}$ denote the set of real numbers. Define scalar multiplication by
$$\alpha x = \alpha \cdot x \quad \text{(the usual multiplication of real numbers)}$$
and define addition, denoted $\oplus$, by
$$x \oplus y = \max(x, y) \quad \text{(the maximum of the two numbers)}$$
Is R a vector space with these operations? Prove your answer.

14. Let Z denote the set of all integers with addition defined in the usual way, and define scalar multiplication, denoted $\circ$, by

$$\alpha \circ k = [\![\alpha]\!] \cdot k \quad \text{for all} \quad k \in Z$$

where $[\![\alpha]\!]$ denotes the greatest integer less than or equal to α. For example,

$$2.25 \circ 4 = [\![2.25]\!] \cdot 4 = 2 \cdot 4 = 8$$

Show that Z, together with these operations, is not a vector space. Which axioms fail to hold?

15. Let S denote the set of all infinite sequences of real numbers with scalar multiplication and addition defined by

$$\alpha\{a_n\} = \{\alpha a_n\}$$
$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

Show that S is a vector space.

16. We can define a one-to-one correspondence between the elements of P_n and $\mathbb{R}^n$ by

$$p(x) = a_1 + a_2x + \cdots + a_nx^{n-1}$$
$$\leftrightarrow (a_1, \ldots, a_n)^T = \mathbf{a}$$

Show that if $p \leftrightarrow \mathbf{a}$ and $q \leftrightarrow \mathbf{b}$, then

(a) $\alpha p \leftrightarrow \alpha\mathbf{a}$ for any scalar α.

(b) $p + q \leftrightarrow \mathbf{a} + \mathbf{b}$.

[In general, two vector spaces are said to be *isomorphic* if their elements can be put into a one-to-one correspondence that is preserved under scalar multiplication and addition as in (a) and (b).]

2 Subspaces

Given a vector space V, it is often possible to form another vector space by taking a subset S of V and using the operations of V. Since V is a vector space, the operations of addition and scalar multiplication always produce another vector in V. For a new system using a subset S of V as its universal set to be a vector space, the set S must be closed under the operations of addition and scalar multiplication. That is, the sum of two elements of S must always be an element of S, and the product of a scalar and an element of S must always be an element of S.

EXAMPLE 1 Let

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_2 = 2x_1 \right\}$$

S is a subset of $\mathbb{R}^2$. If

$$\mathbf{x} = \begin{bmatrix} c \\ 2c \end{bmatrix}$$

is any element of S and α is any scalar, then

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} c \\ 2c \end{bmatrix} = \begin{bmatrix} \alpha c \\ 2\alpha c \end{bmatrix}$$

is also an element of S. If

$$\begin{bmatrix} a \\ 2a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b \\ 2b \end{bmatrix}$$

are any two elements of S, then their sum

$$\begin{bmatrix} a+b \\ 2a+2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2(a+b) \end{bmatrix}$$

is also an element of S. It is easily seen that the mathematical system consisting of the set S (instead of $\mathbb{R}^2$), together with the operations from $\mathbb{R}^2$, is itself a vector space. ■