Setting

$$\begin{aligned}\alpha_3 - \quad \alpha_1 &= a\\ \alpha_2 \quad &= b\\ \alpha_1 + 2\alpha_2 &= c\end{aligned}$$

and solving, we see that $\alpha_1 = c - 2b$, $\alpha_2 = b$, and $\alpha_3 = a + c - 2b$. ■

In Example 11(a), we saw that the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 2, 3)^T$ span $\mathbb{R}^3$. Clearly, $\mathbb{R}^3$ could be spanned with only the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The vector $(1, 2, 3)^T$ is really not necessary. In the next section, we consider the problem of finding minimal spanning sets for a vector space V (i.e., spanning sets that contain the smallest possible number of vectors).

SECTION 2 EXERCISES

1. Determine whether the following sets form subspaces of $\mathbb{R}^2$:

(a) $\{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$

(b) $\{(x_1, x_2)^T \mid x_1 x_2 = 0\}$

(c) $\{(x_1, x_2)^T \mid x_1 = 3x_2\}$

(d) $\{(x_1, x_2)^T \mid |x_1| = |x_2|\}$

(e) $\{(x_1, x_2)^T \mid x_1^2 = x_2^2\}$

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$:

(a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$

(b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$

(c) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$

(d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) The set of all 2×2 diagonal matrices

(b) The set of all 2×2 triangular matrices

(c) The set of all 2×2 lower triangular matrices

(d) The set of all 2×2 matrices A such that $a_{12} = 1$

(e) The set of all 2×2 matrices B such that $b_{11} = 0$

(f) The set of all symmetric 2×2 matrices

(g) The set of all singular 2×2 matrices

4. Determine the null space of each of the following matrices:

(a) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}$

5. Determine whether the following are subspaces of P_4 (be careful!):

(a) The set of polynomials in P_4 of even degree

(b) The set of all polynomials of degree 3

(c) The set of all polynomials $p(x)$ in P_4 such that $p(0) = 0$

(d) The set of all polynomials in P_4 having at least one real root

6. Determine whether the following are subspaces of $C[-1, 1]$:

(a) The set of functions f in $C[-1, 1]$ such that $f(-1) = f(1)$

(b) The set of odd functions in $C[-1, 1]$

(c) The set of continuous nondecreasing functions on $[-1, 1]$

(d) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ and $f(1) = 0$

(e) The set of functions f in $C[-1, 1]$ such that $f(-1) = 0$ or $f(1) = 0$

7. Show that $C^n[a, b]$ is a subspace of $C[a, b]$.

8. Let A be a fixed vector in $\mathbb{R}^{n\times n}$ and let S be the set of all matrices that commute with A; that is,

$$S = \{B \mid AB = BA\}$$

Show that S is a subspace of $\mathbb{R}^{n\times n}$.

9. In each of the following, determine the subspace of $\mathbb{R}^{2\times 2}$ consisting of all matrices that commute with the given matrix:

(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

10. Let A be a particular vector in $\mathbb{R}^{2\times 2}$. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) $S_1 = \{B \in \mathbb{R}^{2\times 2} \mid BA = O\}$

(b) $S_2 = \{B \in \mathbb{R}^{2\times 2} \mid AB \neq BA\}$

(c) $S_3 = \{B \in \mathbb{R}^{2\times 2} \mid AB + B = O\}$

11. Determine whether the following are spanning sets for $\mathbb{R}^2$:

(a) $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ **(b)** $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right\}$

(e) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

12. Which of the sets that follow are spanning sets for $\mathbb{R}^3$? Justify your answers.

(a) $\{(1, 0, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T\}$

(b) $\{(1, 0, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T, (1, 2, 3)^T\}$

(c) $\{(2, 1, -2)^T, (3, 2, -2)^T, (2, 2, 0)^T\}$

(d) $\{(2, 1, -2)^T, (-2, -1, 2)^T, (4, 2, -4)^T\}$

(e) $\{(1, 1, 3)^T, (0, 2, 1)^T\}$

13. Given

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix}$$

(a) Is $\mathbf{x} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

(b) Is $\mathbf{y} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

Prove your answers.

14. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a spanning set for a vector space V.

(a) If we add another vector, $\mathbf{x}_{k+1}$, to the set, will we still have a spanning set? Explain.

(b) If we delete one of the vectors, say $\mathbf{x}_k$, from the set, will we still have a spanning set? Explain.

15. In $\mathbb{R}^{2\times 2}$, let

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbb{R}^{2\times 2}$.

16. Which of the sets that follow are spanning sets for P_3? Justify your answers.

(a) $\{1, x^2, x^2 - 2\}$ **(b)** $\{2, x^2, x, 2x + 3\}$

(c) $\{x + 2, x + 1, x^2 - 1\}$ **(d)** $\{x + 2, x^2 - 1\}$

17. Let S be the vector space of infinite sequences defined in Exercise 15 of Section 1. Let S_0 be the set of $\{a_n\}$ with the property that $a_n \to 0$ as $n \to \infty$. Show that S_0 is a subspace of S.

18. Prove that if S is a subspace of $\mathbb{R}^1$, then either $S = \{\mathbf{0}\}$ or $S = \mathbb{R}^1$.

19. Let A be an $n \times n$ matrix. Prove that the following statements are equivalent:

(a) $N(A) = \{\mathbf{0}\}$. **(b)** A is nonsingular.

(c) For each $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

20. Let U and V be subspaces of a vector space W. Prove that their intersection $U \cap V$ is also a subspace of W.

21. Let S be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_1$ and let T be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_2$. Is $S \cup T$ a subspace of $\mathbb{R}^2$? Explain.

22. Let U and V be subspaces of a vector space W. Define

$$U + V = \{\mathbf{z} \mid \mathbf{z} = \mathbf{u} + \mathbf{v} \text{ where } \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}$$

Show that $U + V$ is a subspace of W.

23. Let S, T, and U be subspaces of a vector space V. We can form new subspaces by using the operations of $\cap$ and $+$ defined in Exercises 20 and 22. When we do arithmetic with numbers, we know that the operation of multiplication distributes over the operation of addition in the sense that

$$a(b + c) = ab + ac$$

It is natural to ask whether similar distributive laws hold for the two operations with subspaces.

(a) Does the intersection operation for subspaces distribute over the addition operation? That is, does

$$S \cap (T + U) = (S \cap T) + (S \cap U)$$

(b) Does the addition operation for subspaces distribute over the intersection operation? That is, does

$$S + (T \cap U) = (S + T) \cap (S + U)$$

3 Linear Independence

In this section, we look more closely at the structure of vector spaces. To begin with, we restrict ourselves to vector spaces that can be generated from a finite set of elements. Each vector in the vector space can be built up from the elements in this generating set using only the operations of addition and scalar multiplication. The generating set is usually referred to as a spanning set. In particular, it is desirable to find a *minimal* spanning set. By minimal, we mean a spanning set with no unnecessary elements (i.e., all the elements in the set are needed in order to span the vector space). To see how to find a minimal spanning set, it is necessary to consider how the vectors in the collection *depend* on each other. Consequently, we introduce the concepts of *linear dependence* and *linear independence*. These simple concepts provide the keys to understanding the structure of vector spaces.

Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Actually, S can be represented in terms of the two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, since the vector $\mathbf{x}_3$ is already in the span of $\mathbf{x}_1$ and $\mathbf{x}_2$; that is,

$$\mathbf{x}_3 = 3\mathbf{x}_1 + 2\mathbf{x}_2 \qquad (1)$$

Any linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ can be reduced to a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\begin{aligned} \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 &= \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3(3\mathbf{x}_1 + 2\mathbf{x}_2) \\ &= (\alpha_1 + 3\alpha_3)\mathbf{x}_1 + (\alpha_2 + 2\alpha_3)\mathbf{x}_2 \end{aligned}$$

Thus,

$$S = \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$$

Equation (1) can be rewritten in the form

$$3\mathbf{x}_1 + 2\mathbf{x}_2 - 1\mathbf{x}_3 = \mathbf{0} \qquad (2)$$

Since the three coefficients in (2) are nonzero, we could solve for any vector in terms of the other two:

$$\mathbf{x}_1 = -\frac{2}{3}\mathbf{x}_2 + \frac{1}{3}\mathbf{x}_3, \quad \mathbf{x}_2 = -\frac{3}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_3, \quad \mathbf{x}_3 = 3\mathbf{x}_1 + 2\mathbf{x}_2$$

It follows that

$$\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \text{Span}(\mathbf{x}_2, \mathbf{x}_3) = \text{Span}(\mathbf{x}_1, \mathbf{x}_3) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$$