EXAMPLE 9 Show that the vectors $1, x, x^2$, and x^3 are linearly independent in $C((-\infty, \infty))$.

Solution

$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12$$

Since $W[1, x, x^2, x^3] \not\equiv 0$, the vectors are linearly independent. ■

SECTION 3 EXERCISES

1. Determine whether the following vectors are linearly independent in $\mathbb{R}^2$:

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2. Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

3. For each of the sets of vectors in Exercise 2, describe geometrically the span of the given vectors.

4. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

5. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be linearly independent vectors in a vector space V.

(a) If we add a vector $\mathbf{x}_{k+1}$ to the collection, will we still have a linearly independent collection of vectors? Explain.

(b) If we delete a vector, say $\mathbf{x}_k$, from the collection, will we still have a linearly independent collection of vectors? Explain.

6. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{y}_1 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_3, \quad \mathbf{y}_3 = \mathbf{x}_3 + \mathbf{x}_1$$

Are $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

7. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{y}_1 = \mathbf{x}_2 - \mathbf{x}_1, \quad \mathbf{y}_2 = \mathbf{x}_3 - \mathbf{x}_2, \quad \mathbf{y}_3 = \mathbf{x}_3 - \mathbf{x}_1$$

Are $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

8. Determine whether the following vectors are linearly independent in P_3:

(a) $1, x^2, x^2 - 2$ (b) $2, x^2, x, 2x + 3$

(c) $x + 2, x + 1, x^2 - 1$ (d) $x + 2, x^2 - 1$

9. For each of the following, show that the given vectors are linearly independent in $C[0, 1]$:

(a) $\cos \pi x, \sin \pi x$ (b) $x^{3/2}, x^{5/2}$

(c) $1, e^x + e^{-x}, e^x - e^{-x}$ (d) e^x, e^{-x}, e^{2x}

10. Determine whether the vectors $\cos x$, 1, $\sin^2(x/2)$ are linearly independent in $C[-\pi, \pi]$.

11. Consider the vectors $\cos(x + \alpha)$ and $\sin x$ in $C[-\pi, \pi]$. For what values of α will the two vectors be linearly dependent? Give a graphical interpretation of your answer.
12. Given the functions $2x$ and $|x|$, show that
    (a) these two vectors are linearly independent in $C[-1, 1]$.
    (b) the vectors are linearly dependent in $C[0, 1]$.
13. Prove that any finite set of vectors that contains the zero vector must be linearly dependent.
14. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors in a vector space V. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of the vectors is a scalar multiple of the other.
15. Prove that any nonempty subset of a linearly independent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is also linearly independent.
16. Let A be an $m \times n$ matrix. Show that if A has linearly independent column vectors, then $N(A) = \{\mathbf{0}\}$.
    [*Hint*: For any $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$.]
17. Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be linearly independent vectors in $\mathbb{R}^n$, and let A be a nonsingular $n \times n$ matrix. Define $\mathbf{y}_i = A\mathbf{x}_i$ for $i = 1, \ldots, k$. Show that $\mathbf{y}_1, \ldots, \mathbf{y}_k$ are linearly independent.
18. Let A be a 3×3 matrix and let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be vectors in $\mathbb{R}^3$. Show that if the vectors

$$\mathbf{y}_1 = A\mathbf{x}_1, \quad \mathbf{y}_2 = A\mathbf{x}_2, \quad \mathbf{y}_3 = A\mathbf{x}_3$$

    are linearly independent, then the matrix A must be nonsingular and the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ must be linearly independent.
19. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for the vector space V, and let $\mathbf{v}$ be any other vector in V. Show that $\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.
20. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent vectors in a vector space V. Show that $\mathbf{v}_2, \ldots, \mathbf{v}_n$ cannot span V.

4 Basis and Dimension

In Section 3, we showed that a spanning set for a vector space is minimal if its elements are linearly independent. The elements of a minimal spanning set form the basic building blocks for the whole vector space, and consequently, we say that they form a *basis* for the vector space.

Definition The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a **basis** for a vector space V if and only if

(i) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

(ii) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V.

EXAMPLE 1 The *standard basis* for $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; however, there are many bases that we could choose for $\mathbb{R}^3$. For example,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are both bases for $\mathbb{R}^3$. We will see shortly that any basis for $\mathbb{R}^3$ must have exactly three elements.