EXAMPLE 13 Let $D: P_3 \to P_3$ be the differentiation operator, defined by

$$D(p(x)) = p'(x)$$

The kernel of D consists of all polynomials of degree 0. Thus $\ker(D) = P_1$. The derivative of any polynomial in P_3 will be a polynomial of degree 1 or less. Conversely, any polynomial in P_2 will have antiderivatives in P_3, so each polynomial in P_2 will be the image of polynomials in P_3 under the operator D. It then follows that $D(P_3) = P_2$. ■

SECTION 1 EXERCISES

1. Show that each of the following are linear operators on $\mathbb{R}^2$. Describe geometrically what each linear transformation accomplishes.

(a) $L(\mathbf{x}) = (-x_1, x_2)^T$ (b) $L(\mathbf{x}) = -\mathbf{x}$

(c) $L(\mathbf{x}) = (x_2, x_1)^T$ (d) $L(\mathbf{x}) = \frac{1}{2}\mathbf{x}$

(e) $L(\mathbf{x}) = x_2\mathbf{e}_2$

2. Let L be the linear operator on $\mathbb{R}^2$ defined by

$$L(\mathbf{x}) = (x_1 \cos\alpha - x_2 \sin\alpha,\ x_1 \sin\alpha + x_2 \cos\alpha)^T$$

Express x_1, x_2, and $L(\mathbf{x})$ in terms of polar coordinates. Describe geometrically the effect of the linear transformation.

3. Let $\mathbf{a}$ be a fixed nonzero vector in $\mathbb{R}^2$. A mapping of the form

$$L(\mathbf{x}) = \mathbf{x} + \mathbf{a}$$

is called a *translation*. Show that a translation is not a linear operator. Illustrate geometrically the effect of a translation.

4. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L((1, 2)^T) = (-2, 3)^T$$

and

$$L((1, -1)^T) = (5, 2)^T$$

find the value of $L((7, 5)^T)$.

5. Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$:

(a) $L(\mathbf{x}) = (x_2, x_3)^T$ (b) $L(\mathbf{x}) = (0, 0)^T$

(c) $L(\mathbf{x}) = (1 + x_1, x_2)^T$

(d) $L(\mathbf{x}) = (x_3, x_1 + x_2)^T$

6. Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$:

(a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$

(b) $L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T$

(c) $L(\mathbf{x}) = (x_1, 0, 0)^T$

(d) $L(\mathbf{x}) = (x_1, x_2, x_1^2 + x_2^2)^T$

7. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$:

(a) $L(A) = 2A$ (b) $L(A) = A^T$

(c) $L(A) = A + I$ (d) $L(A) = A - A^T$

8. Let C be a fixed $n \times n$ matrix. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$:

(a) $L(A) = CA + AC$ (b) $L(A) = C^2A$

(c) $L(A) = A^2C$

9. Determine whether the following are linear transformations from P_2 to P_3:

(a) $L(p(x)) = xp(x)$

(b) $L(p(x)) = x^2 + p(x)$

(c) $L(p(x)) = p(x) + xp(x) + x^2p'(x)$

10. For each $f \in C[0, 1]$, define $L(f) = F$, where

$$F(x) = \int_0^x f(t)\,dt \qquad 0 \le x \le 1$$

Show that L is a linear operator on $C[0, 1]$ and then find $L(e^x)$ and $L(x^2)$.

11. Determine whether the following are linear transformations from $C[0, 1]$ into $\mathbb{R}^1$:

(a) $L(f) = f(0)$ (b) $L(f) = |f(0)|$

(c) $L(f) = [f(0) + f(1)]/2$

(d) $L(f) = \left\{\int_0^1 [f(x)]^2\,dx\right\}^{1/2}$

12. Use mathematical induction to prove that if L is a linear transformation from V to W, then

$$L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_n L(\mathbf{v}_n)$$

13. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let L_1 and L_2 be two linear transformations mapping V into a vector space W. Show that if

$$L_1(\mathbf{v}_i) = L_2(\mathbf{v}_i)$$

for each $i = 1, \ldots, n$, then $L_1 = L_2$ [i.e., show that $L_1(\mathbf{v}) = L_2(\mathbf{v})$ for all $\mathbf{v} \in V$].

14. Let L be a linear operator on $\mathbb{R}^1$ and let $a = L(1)$. Show that $L(x) = ax$ for all $x \in \mathbb{R}^1$.

15. Let L be a linear operator on a vector space V. Define L^n, $n \geq 1$, recursively by

$$L^1 = L$$
$$L^{k+1}(\mathbf{v}) = L(L^k(\mathbf{v})) \qquad \text{for all } \mathbf{v} \in V$$

Show that L^n is a linear operator on V for each $n \geq 1$.

16. Let $L_1 : U \rightarrow V$ and $L_2 : V \rightarrow W$ be linear transformations, and let $L = L_2 \circ L_1$ be the mapping defined by

$$L(\mathbf{u}) = L_2(L_1(\mathbf{u}))$$

for each $\mathbf{u} \in U$. Show that L is a linear transformation mapping U into W.

17. Determine the kernel and range of each of the following linear operators on $\mathbb{R}^3$:

(a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$

(b) $L(\mathbf{x}) = (x_1, x_2, 0)^T$

(c) $L(\mathbf{x}) = (x_1, x_1, x_1)^T$

18. Let S be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. For each linear operator L in Exercise 17, find $L(S)$.

19. Find the kernel and range of each of the following linear operators on P_3:

(a) $L(p(x)) = xp'(x)$

(b) $L(p(x)) = p(x) - p'(x)$

(c) $L(p(x)) = p(0)x + p(1)$

20. Let $L : V \rightarrow W$ be a linear transformation, and let T be a subspace of W. The *inverse image* of T, denoted $L^{-1}(T)$, is defined by

$$L^{-1}(T) = \{\mathbf{v} \in V \mid L(\mathbf{v}) \in T\}$$

Show that $L^{-1}(T)$ is a subspace of V.

21. A linear transformation $L : V \rightarrow W$ is said to be *one-to-one* if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (i.e., no two distinct vectors $\mathbf{v}_1, \mathbf{v}_2$ in V get mapped into the same vector $\mathbf{w} \in W$). Show that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}_V\}$.

22. A linear transformation $L : V \rightarrow W$ is said to map V *onto* W if $L(V) = W$. Show that the linear transformation L defined by

$$L(\mathbf{x}) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$$

maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

23. Which of the operators defined in Exercise 17 are one-to-one? Which map $\mathbb{R}^3$ onto $\mathbb{R}^3$?

24. Let A be a 2×2 matrix, and let L_A be the linear operator defined by

$$L_A(\mathbf{x}) = A\mathbf{x}$$

Show that

(a) L_A maps $\mathbb{R}^2$ onto the column space of A.

(b) if A is nonsingular, then L_A maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

25. Let D be the differentiation operator on P_3, and let

$$S = \{p \in P_3 \mid p(0) = 0\}$$

Show that

(a) D maps P_3 onto the subspace P_2, but $D : P_3 \rightarrow P_2$ is not one-to-one.

(b) $D : S \rightarrow P_3$ is one-to-one but not onto.

2 Matrix Representations of Linear Transformations

In Section 1, it was shown that each $m \times n$ matrix A defines a linear transformation L_A from $\mathbb{R}^n$ to $\mathbb{R}^m$, where

$$L_A(\mathbf{x}) = A\mathbf{x}$$

for each $\mathbf{x} \in \mathbb{R}^n$. In this section, we will see that, for each linear transformation L mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, there is an $m \times n$ matrix A such that

$$L(\mathbf{x}) = A\mathbf{x}$$