PY. The effect of the yaw on the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ is to rotate them to the new directions $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$. So the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ will define the directions of the x, y, and z axes when we do the pitch. The desired pitch transformation is then a rotation about the new y-axis (i.e., the axis in the direction of the vector $\mathbf{y}_2$). The vectors $\mathbf{y}_1$ and $\mathbf{y}_3$ form a plane, and when the pitch is applied, they are both rotated by an angle v in that plane. The vector $\mathbf{y}_2$ will remain unaffected by the pitch, since it lies on the axis of rotation. Thus, the composite transformation L has the following effect on the standard basis vectors:

$$\begin{aligned}
\mathbf{e}_1 &\overset{\text{yaw}}{\rightarrow} \mathbf{y}_1 \overset{\text{pitch}}{\rightarrow} \cos v\, \mathbf{y}_1 + \sin v\, \mathbf{y}_3 \\
\mathbf{e}_2 &\overset{\text{yaw}}{\rightarrow} \mathbf{y}_2 \overset{\text{pitch}}{\rightarrow} \mathbf{y}_2 \\
\mathbf{e}_3 &\overset{\text{yaw}}{\rightarrow} \mathbf{y}_3 \overset{\text{pitch}}{\rightarrow} -\sin v\, \mathbf{y}_1 + \cos v\, \mathbf{y}_3
\end{aligned}$$

The images of the standard basis vectors form the columns of the matrix representing the composite transformation:

$$\begin{aligned}
(\cos v\, \mathbf{y}_1 + \sin v\, \mathbf{y}_3, \mathbf{y}_2, -\sin v\, \mathbf{y}_1 + \cos v\, \mathbf{y}_3) &= (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) \begin{Bmatrix} \cos v & 0 & -\sin v \\ 0 & 1 & 0 \\ \sin v & 0 & \cos v \end{Bmatrix} \\
&= YP
\end{aligned}$$

It follows that matrix representation of the composite is a product of the two individual matrices representing the yaw and the pitch, but the product must be taken in the reverse order, with the yaw matrix Y on the left and the pitch matrix P on the right. Similarly, for a composite transformation of a yaw with angle u, followed by a pitch with angle v, and then a roll with angle w, the matrix representation of the composite transformation would be the product YPR.

SECTION 2 EXERCISES

1. Refer to Exercise 1 of Section 1. For each linear transformation L, find the standard matrix representation of L.

2. For each of the following linear transformations L mapping $\mathbb{R}^3$ into $\mathbb{R}^2$, find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$:

(a) $L((x_1, x_2, x_3)^T) = (x_1 + x_2, 0)^T$

(b) $L((x_1, x_2, x_3)^T) = (x_1, x_2)^T$

(c) $L((x_1, x_2, x_3)^T) = (x_2 - x_1, x_3 - x_2)^T$

3. For each of the following linear operators L on $\mathbb{R}^3$, find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$:

(a) $L((x_1, x_2, x_3)^T) = (x_3, x_2, x_1)^T$

(b) $L((x_1, x_2, x_3)^T) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$

(c) $L((x_1, x_2, x_3)^T) = (2x_3, x_2 + 3x_1, 2x_1 - x_3)^T$

4. Let L be the linear operator on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{Bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{Bmatrix}$$

Determine the standard matrix representation A of L, and use A to find $L(\mathbf{x})$ for each of the following vectors $\mathbf{x}$:

(a) $\mathbf{x} = (1, 1, 1)^T$ **(b)** $\mathbf{x} = (2, 1, 1)^T$

(c) $\mathbf{x} = (-5, 3, 2)^T$

5. Find the standard matrix representation for each of the following linear operators:

(a) L is the linear operator that rotates each $\mathbf{x}$ in $\mathbb{R}^2$ by 45° in the clockwise direction.

(b) L is the linear operator that reflects each vector $\mathbf{x}$ in $\mathbb{R}^2$ about the x_1-axis and then rotates it 90° in the counterclockwise direction.

(c) L doubles the length of $\mathbf{x}$ and then rotates it 30° in the counterclockwise direction.

(d) L reflects each vector $\mathbf{x}$ about the line $x_2 = x_1$ and then projects it onto the x_1-axis.

6. Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3$$

Find the matrix A representing L with respect to the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

7. Let

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and let $\mathcal{I}$ be the identity operator on $\mathbb{R}^3$.

(a) Find the coordinates of $\mathcal{I}(\mathbf{e}_1)$, $\mathcal{I}(\mathbf{e}_2)$, and $\mathcal{I}(\mathbf{e}_3)$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

(b) Find a matrix A such that $A\mathbf{x}$ is the coordinate vector of $\mathbf{x}$ with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

8. Let $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ be defined as in Exercise 7, and let L be the linear operator on $\mathbb{R}^3$ defined by

$$L(c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3)$$
$$= (c_1 + c_2 + c_3)\mathbf{y}_1 + (2c_1 + c_3)\mathbf{y}_2 - (2c_2 + c_3)\mathbf{y}_3$$

(a) Find a matrix representing L with respect to the ordered basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

(b) For each of the following, write the vector $\mathbf{x}$ as a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ and use the matrix from part (a) to determine $L(\mathbf{x})$:

(i) $\mathbf{x} = (7, 5, 2)^T$ (ii) $\mathbf{x} = (3, 2, 1)^T$
(iii) $\mathbf{x} = (1, 2, 3)^T$

9. Let

$$R = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The column vectors of R represent the homogeneous coordinates of points in the plane.

(a) Draw the figure whose vertices correspond to the column vectors of R. What type of figure is it?

(b) For each of the following choices of A, sketch the graph of the figure represented by AR and describe geometrically the effect of the linear transformation:

(i) $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

10. For each of the following linear operators on $\mathbb{R}^2$, find the matrix representation of the transformation with respect to the homogeneous coordinate system:

(a) The transformation L that rotates each vector by 120° in the counterclockwise direction

(b) The transformation L that translates each point 3 units to the left and 5 units up

(c) The transformation L that contracts each vector by a factor of one-third

(d) The transformation that reflects a vector about the y-axis and then translates it up 2 units

11. Determine the matrix representation of each of the following composite transformations:

(a) A yaw of 90°, followed by a pitch of 90°

(b) A pitch of 90°, followed by a yaw of 90°

(c) A pitch of 45°, followed by a roll of −90°

(d) A roll of −90°, followed by a pitch of 45°

(e) A yaw of 45°, followed by a pitch of −90° and then a roll of −45°

(f) A roll of −45°, followed by a pitch of −90° and then a yaw of 45°

12. Let Y, P, and R be the yaw, pitch, and roll matrices given in equations (1), (2), and (3), and let $Q = YPR$.

(a) Show that Y, P, and R all have determinants equal to 1.

(b) The matrix Y represents a yaw with angle u. The inverse transformation should be a yaw with angle $-u$. Show that the matrix representation of the inverse transformation is Y^T and that $Y^T = Y^{-1}$.

(c) Show that Q is nonsingular, and express Q^{-1} in terms of the transposes of Y, P, and R.