TABLE 3

SPEARMAN'S CORRELATION MATRIX

	Classics	French	English	Math	Discrim.	Music
Classics	1	0.83	0.78	0.70	0.66	0.63
French	0.83	1	0.67	0.67	0.65	0.57
English	0.78	0.67	1	0.64	0.54	0.51
Math	0.70	0.67	0.64	1	0.45	0.51
Discrim.	0.66	0.65	0.54	0.45	1	0.40
Music	0.63	0.57	0.51	0.51	0.40	1

The hypothetical factors can be isolated mathematically using a method known as *principal component analysis*. The basic idea is to form a matrix X of deviations from the mean and then factor it into a product UW, where the columns of U correspond to the hypothetical factors. While in practice the columns of X are positively correlated, the hypothetical factors should be uncorrelated. Thus the column vectors of U should be mutually orthogonal (i.e., $\mathbf{u}_i^T\mathbf{u}_j = 0$ whenever $i \neq j$). The entries in each column of U measure how well the individual students exhibit the particular intellectual ability represented by that column. The matrix W measures to what extent each test depends on the hypothetical factors.

The construction of the principal component vectors relies on the covariance matrix $S = \frac{1}{n-1}X^TX$. Since it depends on the *eigenvalues* and *eigenvectors* of the S, we will defer the details of the method until Chapter 6. In Section 5 of Chapter 6 we will revisit this application and learn an important factorization called the *singular value decomposition*, which is the main tool of principal component analysis.

REFERENCES

1. Spearman, C., 'General Intelligence', objectively determined and measured, *American Journal of Psychology*, **15**, 1904.
2. Hotelling, H., Analysis of a complex of statistical variables in principal components, *Journal of Educational Psychology*, **26**, 1933.
3. Maxwell, A. E., *Multivariate Analysis in Behavioral Research*, Chapman and Hall, London, 1977.

EXERCISES

1. Find the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$ in each of the following.

(a) $\mathbf{v} = (2, 1, 3)^T$, $\mathbf{w} = (6, 3, 9)^T$ (b) $\mathbf{v} = (2, -3)^T$, $\mathbf{w} = (3, 2)^T$

(c) $\mathbf{v} = (4, 1)^T$, $\mathbf{w} = (3, 2)^T$ (d) $\mathbf{v} = (-2, 3, 1)^T$, $\mathbf{w} = (1, 2, 4)^T$

2. For each of the pairs of vectors in Exercise 1, find the scalar projection of $\mathbf{v}$ onto $\mathbf{w}$. Also find the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.

3. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$ and verify that $\mathbf{p}$ and $\mathbf{x} - \mathbf{p}$ are orthogonal.

   (a) $\mathbf{x} = (3, 4)^T,\ \mathbf{y} = (1, 0)^T$ (b) $\mathbf{x} = (3, 5)^T,\ \mathbf{y} = (1, 1)^T$
   (c) $\mathbf{x} = (2, 4, 3)^T,\ \mathbf{y} = (1, 1, 1)^T$ (d) $\mathbf{x} = (2, -5, 4)^T,\ \mathbf{y} = (1, 2, -1)^T$

4. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent vectors in R^2. If $\|\mathbf{x}\| = 2$ and $\|\mathbf{y}\| = 3$, what, if anything, can we conclude about the possible values of $|\mathbf{x}^T\mathbf{y}|$?

5. Find the point on the line $y = 2x$ that is closest to the point $(5, 2)$.

6. Find the point on the line $y = 2x + 1$ that is closest to the point $(5, 2)$.

7. Find the distance from the point $(1, 2)$ to the line $4x - 3y = 0$.

8. In each of the following, find the equation of the plane normal to the given vector $\mathbf{N}$ and passing through the point P_0.

   (a) $\mathbf{N} = (2, 4, 3)^T,\ P_0 = (0, 0, 0)$ (b) $\mathbf{N} = (-3, 6, 2)^T,\ P_0 = (4, 2, -5)$
   (c) $\mathbf{N} = (0, 0, 1)^T,\ P_0 = (3, 2, 4)$

9. Find the distance from the point $(1, 1, 1)$ to the plane $2x + 2y + z = 0$.

10. Find the distance from the point $(2, 1, -2)$ to the plane $6(x - 1) + 2(y - 3) + 3(z + 4) = 0$.

11. If $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{y} = (y_1, y_2)^T$, and $\mathbf{z} = (z_1, z_2)^T$ are arbitrary vectors in R^2, prove:

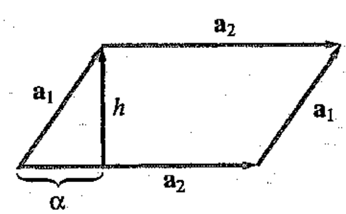
    (a) $\mathbf{x}^T\mathbf{x} \geq 0$ (b) $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$ (c) $\mathbf{x}^T(\mathbf{y} + \mathbf{z}) = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z}$

12. If $\mathbf{u}$ and $\mathbf{v}$ are any vectors in R^2, show that $\|\mathbf{u} + \mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$ and hence $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. When does equality hold? Give a geometric interpretation of the inequality.

13. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be vectors in R^3. If $\mathbf{x}_1 \perp \mathbf{x}_2$ and $\mathbf{x}_2 \perp \mathbf{x}_3$, is it necessarily true that $\mathbf{x}_1 \perp \mathbf{x}_3$? Prove your answer.

14. Let A be a 2×2 matrix with linearly independent column vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. If $\mathbf{a}_1$ and $\mathbf{a}_2$ are used to form a parallelogram P with altitude h (see the figure), show that:

    (a) $h^2\|\mathbf{a}_2\|^2 = \|\mathbf{a}_1\|^2\|\mathbf{a}_2\|^2 - (\mathbf{a}_1^T\mathbf{a}_2)^2$ (b) Area of $P = |\det(A)|$



15. Given

$$\mathbf{x} = \begin{bmatrix} 4 \\ 4 \\ -4 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

(a) Determine the angle between $\mathbf{x}$ and $\mathbf{y}$.

(b) Determine the distance between $\mathbf{x}$ and $\mathbf{y}$.

16. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in R^n and define

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} \quad \text{and} \quad \mathbf{z} = \mathbf{x} - \mathbf{p}$$

(a) Show that $\mathbf{p} \perp \mathbf{z}$. Thus $\mathbf{p}$ is the *vector projection* of $\mathbf{x}$ onto $\mathbf{y}$; that is, $\mathbf{x} = \mathbf{p} + \mathbf{z}$, where $\mathbf{p}$ and $\mathbf{z}$ are orthogonal components of $\mathbf{x}$, and $\mathbf{p}$ is a scalar multiple of $\mathbf{y}$.

(b) If $\|\mathbf{p}\| = 6$ and $\|\mathbf{z}\| = 8$, determine the value of $\|\mathbf{x}\|$.

17. Use the database matrix U from Application 1 and search for the key words *orthogonality, spaces, vector*, only this time give the key word *orthogonality* twice the weight of the other two key words. Which of the eight modules best matches the search criteria? [*Hint:* Form the search vector using the weights 2, 1, 1 in the rows corresponding to the search words and then scale the vector to make it a unit vector.]

18. Five students in an elementary school take aptitude tests in English, Mathematics, and Science. Their scores are given in the table. Determine the correlation matrix and describe how the three sets of scores are correlated.

	Scores		
Student	**English**	**Mathematics**	**Science**
S1	61	53	53
S2	63	73	78
S3	78	61	82
S4	65	84	96
S5	63	59	71
Average	66	66	76

2 ORTHOGONAL SUBSPACES

Let A be an $m \times n$ matrix and let $\mathbf{x} \in N(A)$, the nullspace of A. Since $A\mathbf{x} = \mathbf{0}$, we have

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0 \tag{1}$$