from an inner product, the Pythagorean Law will not hold. For example,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

are orthogonal; however,

$$\|\mathbf{x}_1\|_\infty^2 + \|\mathbf{x}_2\|_\infty^2 = 4 + 16 = 20$$

while

$$\|\mathbf{x}_1 + \mathbf{x}_2\|_\infty^2 = 16$$

On the other hand, if $\|\cdot\|_2$ is used, then

$$\|\mathbf{x}_1\|_2^2 + \|\mathbf{x}_2\|_2^2 = 5 + 20 = 25 = \|\mathbf{x}_1 + \mathbf{x}_2\|_2^2$$

EXAMPLE 5. Let $\mathbf{x}$ be the vector $(4, -5, 3)^T$ in R^3. Compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$.

$$\|\mathbf{x}\|_1 = |4| + |-5| + |3| = 12$$
$$\|\mathbf{x}\|_2 = \sqrt{16 + 25 + 9} = 5\sqrt{2}$$
$$\|\mathbf{x}\|_\infty = \max(|4|, |-5|, |3|) = 5$$

◀

It is also possible to define different matrix norms for $R^{m\times n}$. In Chapter 7 we will study other types of matrix norms that are useful in determining the sensitivity of linear systems.

In general, a norm provides a way of measuring the distance between vectors.

▶ **DEFINITION** Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in a normed linear space. The distance between $\mathbf{x}$ and $\mathbf{y}$ is defined to be the number $\|\mathbf{y} - \mathbf{x}\|$. ◀

Many applications involve finding a unique closest vector in a subspace S to a given vector $\mathbf{v}$ in a vector space V. If the norm used for V is derived from an inner product, then the closest vector can be computed as a vector projection of $\mathbf{v}$ onto the subspace S. This type of approximation problem is discussed further in the next section.

EXERCISES

1. Let $\mathbf{x} = (-1, -1, 1, 1)^T$ and $\mathbf{y} = (1, 1, 5, -3)^T$. Show that $\mathbf{x} \perp \mathbf{y}$. Calculate $\|\mathbf{x}\|_2$, $\|\mathbf{y}\|_2$, $\|\mathbf{x} + \mathbf{y}\|_2$ and verify that the Pythagorean Law holds.

2. Given $\mathbf{x} = (1, 1, 1, 1)^T$ and $\mathbf{y} = (8, 2, 2, 0)^T$:

(a) Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.
(b) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.
(c) Verify that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.
(d) Compute $\|\mathbf{x} - \mathbf{p}\|_2$, $\|\mathbf{p}\|_2$, $\|\mathbf{x}\|_2$ and verify that the Pythagorean Law is satisfied.

3. Use equation (1) with weight vector $\mathbf{w} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^T$ to define an inner product for R^3 and let $\mathbf{x} = (1, 1, 1)^T$ and $\mathbf{y} = (-5, 1, 3)^T$.

   (a) Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal with respect to this inner product.
   (b) Compute the values of $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ with respect to this inner product.

4. Given

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

   Determine the value of each of the following.
   (a) $\langle A, B\rangle$ (b) $\|A\|_F$ (c) $\|B\|_F$ (d) $\|A + B\|_F$

5. Show that equation (2) defines an inner product on $R^{m\times n}$.

6. Show that the inner product defined by equation (3) satisfies the last two conditions of the definition of an inner product.

7. In $C[0, 1]$ with inner product defined by (3), compute:
   (a) $\langle e^x, e^{-x}\rangle$ (b) $\langle x, \sin \pi x\rangle$ (c) $\langle x^2, x^3\rangle$

8. In $C[0, 1]$, with inner product defined by (3), consider the vectors 1 and x.

   (a) Find the angle θ between 1 and x.
   (b) Determine the vector projection $\mathbf{p}$ of 1 onto x and verify that $1 - \mathbf{p}$ is orthogonal to $\mathbf{p}$.
   (c) Compute $\|1 - \mathbf{p}\|$, $\|\mathbf{p}\|$, $\|1\|$ and verify that the Pythagorean Law holds.

9. In $C[-\pi, \pi]$ with inner product defined by (6), show that $\cos mx$ and $\sin nx$ are orthogonal and that both are unit vectors. Determine the distance between the two vectors.

10. Show that the functions x and x^2 are orthogonal in P_5 with inner product defined by (5), where $x_i = (i - 3)/2$ for $i = 1, \ldots, 5$.

11. In P_5 with inner product as in Exercise 10 and norm defined by

$$\|p\| = \sqrt{\langle p, p\rangle} = \left\{\sum_{i=1}^{5} [p(x_i)]^2\right\}^{1/2}$$

    compute:
    (a) $\|x\|$ (b) $\|x^2\|$ (c) The distance between x and x^2

12. If V is an inner product space, show that

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

    satisfies the first two conditions in the definition of a norm.

13. Show that

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

defines a norm on R^n.

14. Show that

$$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

defines a norm on R^n.

15. Compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ for each of the following vectors in R^3.

(a) $\mathbf{x} = (-3, 4, 0)^T$ (b) $\mathbf{x} = (-1, -1, 2)^T$ (c) $\mathbf{x} = (1, 1, 1)^T$

16. Let $\mathbf{x} = (5, 2, 4)^T$ and $\mathbf{y} = (3, 3, 2)^T$. Compute $\|\mathbf{x}-\mathbf{y}\|_1$, $\|\mathbf{x}-\mathbf{y}\|_2$, and $\|\mathbf{x}-\mathbf{y}\|_\infty$. Under which norm are the two vectors closest together? Under which norm are they farthest apart?

17. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in an inner product space. Show that if $\mathbf{x} \perp \mathbf{y}$, then the distance between $\mathbf{x}$ and $\mathbf{y}$ is

$$\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)^{1/2}$$

18. In R^n with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$$

derive a formula for the distance between two vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$.

19. Let $\mathbf{x} \in R^n$. Show that $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$.

20. Let $\mathbf{x} \in R^2$. Show that $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$. [*Hint:* Write $\mathbf{x}$ in the form $x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ and use the triangle inequality.]

21. Give an example of a nonzero vector $\mathbf{x} \in R^2$ for which

$$\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_2 = \|\mathbf{x}\|_1$$

22. Show that in any vector space with a norm

$$\|-\mathbf{v}\| = \|\mathbf{v}\|$$

23. Show that for any $\mathbf{u}$ and $\mathbf{v}$ in a normed vector space

$$\|\mathbf{u} + \mathbf{v}\| \ge \left| \|\mathbf{u}\| - \|\mathbf{v}\| \right|$$

24. Prove that for any $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

Give a geometric interpretation of this result for the vector space R^2.

25. The result of Exercise 24 is not valid for norms other than the norm derived from the inner product. Give an example of this in R^2 using $\|\cdot\|_1$.