DFT algorithm requires $8N^2 = 8 \cdot 2^{40}$ operations, that is, approximately 8.8 trillion operations. On the other hand, the FFT algorithm requires only $100N = 100 \cdot 2^{20}$, or approximately 100 million, operations. The ratio of these two operations counts is

$$r = \frac{8N^2}{5N \log_2 N} = 0.08 \cdot 1{,}048{,}576 = 83{,}886$$

In this case, the FFT algorithm is approximately 84,000 times faster than the DFT algorithm.

SECTION 5 EXERCISES

1. Which of the following sets of vectors form an orthonormal basis for $\mathbb{R}^2$?

(a) $\{(1,0)^T, (0,1)^T\}$

(b) $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$

(c) $\{(1,-1)^T, (1,1)^T\}$

(d) $\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\}$

2. Let

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Let $\mathbf{x} = (1, 1, 1)^T$. Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ using Theorem 5.2 and use Parseval's formula to compute $\|\mathbf{x}\|$.

3. Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ of Exercise 2. Let $\mathbf{x} = (1, 2, 2)^T$. Find the projection $\mathbf{p}$ of $\mathbf{x}$ onto S. Show that $(\mathbf{p}-\mathbf{x}) \perp \mathbf{u}_2$ and $(\mathbf{p}-\mathbf{x}) \perp \mathbf{u}_3$.

4. Let θ be a fixed real number and let

$$\mathbf{x}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(a) Show that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

(b) Given a vector $\mathbf{y}$ in $\mathbb{R}^2$, write it as a linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

(c) Verify that

$$c_1^2 + c_2^2 = \|\mathbf{y}\|^2 = y_1^2 + y_2^2$$

5. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for $\mathbb{R}^2$ and let $\mathbf{u}$ be a unit vector in $\mathbb{R}^2$. If $\mathbf{u}^T\mathbf{u}_1 = \frac{1}{2}$, determine the value of $|\mathbf{u}^T\mathbf{u}_2|$.

6. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V and let

$$\mathbf{u} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3 \quad \text{and} \quad \mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3$$

Determine the value of each of the following:

(a) $\langle \mathbf{u}, \mathbf{v} \rangle$ (b) $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$

(c) The angle θ between $\mathbf{u}$ and $\mathbf{v}$

7. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ is a vector with the properties $\|\mathbf{x}\| = 5$, $\langle \mathbf{u}_1, \mathbf{x} \rangle = 4$, and $\mathbf{x} \perp \mathbf{u}_2$, then what are the possible values of c_1, c_2, and c_3?

8. The functions $\cos x$ and $\sin x$ form an orthonormal set in $C[-\pi, \pi]$. If

$$f(x) = 3\cos x + 2\sin x \quad \text{and} \quad g(x) = \cos x - \sin x$$

use Corollary 5.3 to determine the value of

$$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$$

9. The set

$$S = \left\{\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \cos 3x, \cos 4x\right\}$$

is an orthonormal set of vectors in $C[-\pi, \pi]$ with inner product defined by (2).

(a) Use trigonometric identities to write the function $\sin^4 x$ as a linear combination of elements of S.

(b) Use part (a) and Theorem 5.2 to find the values of the following integrals:

(i) $\int_{-\pi}^{\pi} \sin^4 x \cos x \, dx$

(ii) $\int_{-\pi}^{\pi} \sin^4 x \cos 2x \, dx$

(iii) $\int_{-\pi}^{\pi} \sin^4 x \cos 3x \, dx$

(iv) $\int_{-\pi}^{\pi} \sin^4 x \cos 4x \, dx$

10. Write out the Fourier matrix F_8. Show that $F_8 P_8$ can be partitioned into block form:

$$\begin{Bmatrix} F_4 & D_4 F_4 \\ F_4 & -D_4 F_4 \end{Bmatrix}$$

11. Prove that the transpose of an orthogonal matrix is an orthogonal matrix.

12. If Q is an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then how does the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ compare with the angle between $\mathbf{x}$ and $\mathbf{y}$? Prove your answer.

13. Let Q be an $n \times n$ orthogonal matrix. Use mathematical induction to prove each of the following:

(a) $(Q^m)^{-1} = (Q^T)^m = (Q^m)^T$ for any positive integer m.

(b) $\|Q^m\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$.

14. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is both orthogonal and symmetric and hence is its own inverse.

15. Let Q be an orthogonal matrix and let $d = \det(Q)$. Show that $|d| = 1$.

16. Show that the product of two orthogonal matrices is also an orthogonal matrix. Is the product of two permutation matrices a permutation matrix? Explain.

17. How many $n \times n$ permutation matrices are there?

18. Show that if P is a symmetric permutation matrix, then $P^{2k} = I$ and $P^{2k+1} = P$.

19. Show that if U is an $n \times n$ orthogonal matrix, then

$$\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T = I$$

20. Use mathematical induction to show that if an $n \times n$ matrix Q is both upper triangular and orthogonal, then $\mathbf{q}_j = \pm\mathbf{e}_j$, $j = 1, \ldots, n$.

21. Let

$$A = \begin{Bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix}$$

(a) Show that the column vectors of A form an orthonormal set in $\mathbb{R}^4$.

(b) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$ for each of the following choices of $\mathbf{b}$:

(i) $\mathbf{b} = (4, 0, 0, 0)^T$

(ii) $\mathbf{b} = (1, 2, 3, 4)^T$

(iii) $\mathbf{b} = (1, 1, 2, 2)^T$

22. Let A be the matrix given in Exercise 21.

(a) Find the projection matrix P that projects vectors in $\mathbb{R}^4$ onto $R(A)$.

(b) For each of your solutions $\mathbf{x}$ to Exercise 21(b), compute $A\mathbf{x}$ and compare it with $P\mathbf{b}$.

23. Let A be the matrix given in Exercise 21.

(a) Find an orthonormal basis for $N(A^T)$.

(b) Determine the projection matrix Q that projects vectors in $\mathbb{R}^4$ onto $N(A^T)$.

24. Let A be an $m \times n$ matrix, let P be the projection matrix that projects vectors in $\mathbb{R}^m$ onto $R(A)$, and let Q be the projection matrix that projects vectors in $\mathbb{R}^n$ onto $R(A^T)$. Show that

(a) $I - P$ is the projection matrix from $\mathbb{R}^m$ onto $N(A^T)$.

(b) $I - Q$ is the projection matrix from $\mathbb{R}^n$ onto $N(A)$.

25. Let P be the projection matrix corresponding to a subspace S of $\mathbb{R}^m$. Show that

(a) $P^2 = P$ (b) $P^T = P$

26. Let A be an $m \times n$ matrix whose column vectors are mutually orthogonal, and let $\mathbf{b} \in \mathbb{R}^m$. Show that if $\mathbf{y}$ is the least squares solution of the system $A\mathbf{x} = \mathbf{b}$, then

$$y_i = \frac{\mathbf{b}^T\mathbf{a}_i}{\mathbf{a}_i^T\mathbf{a}_i} \qquad i = 1, \ldots, n$$

27. Let $\mathbf{v}$ be a vector in an inner product space V and let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto an n-dimensional subspace S of V. Show that $\|\mathbf{p}\| \leq \|\mathbf{v}\|$. Under what conditions does equality occur?

28. Let $\mathbf{v}$ be a vector in an inner product space V and let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto an n-dimensional subspace S of V. Show that $\|\mathbf{p}\|^2 = \langle\mathbf{p}, \mathbf{v}\rangle$.

29. Consider the vector space $C[-1, 1]$ with inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

and norm

$$\|f\| = (\langle f, f\rangle)^{1/2}$$

(a) Show that the vectors 1 and x are orthogonal.

(b) Compute $\|1\|$ and $\|x\|$.

(c) Find the best least squares approximation to $x^{1/3}$ on $[-1, 1]$ by a linear function $l(x) = c_1 1 + c_2 x$.

(d) Sketch the graphs of $x^{1/3}$ and $l(x)$ on $[-1, 1]$.

30. Consider the inner product space $C[0, 1]$ with inner product defined by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$$

Let S be the subspace spanned by the vectors 1 and $2x - 1$.

(a) Show that 1 and $2x - 1$ are orthogonal.

(b) Determine $\|1\|$ and $\|2x - 1\|$.

(c) Find the best least squares approximation to $\sqrt{x}$ by a function from the subspace S.

31. Let

$$S = \{1/\sqrt{2}, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx\}$$

Show that S is an orthonormal set in $C[-\pi, \pi]$ with inner product defined by (2).

32. Find the best least squares approximation to $f(x) = |x|$ on $[-\pi, \pi]$ by a trigonometric polynomial of degree less than or equal to 2.

33. Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n\}$ be an orthonormal basis for an inner product space V. Let S_1 be the subspace of V spanned by $\mathbf{x}_1, \dots, \mathbf{x}_k$, and let S_2 be the subspace spanned by $\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \dots, \mathbf{x}_n$. Show that $S_1 \perp S_2$.

34. Let $\mathbf{x}$ be an element of the inner product space V in Exercise 33, and let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the projections of $\mathbf{x}$ onto S_1 and S_2, respectively. Show that

(a) $\mathbf{x} = \mathbf{p}_1 + \mathbf{p}_2$.

(b) if $\mathbf{x} \in S_1^\perp$, then $\mathbf{p}_1 = \mathbf{0}$ and hence $S^\perp = S_2$.

35. Let S be a subspace of an inner product space V. Let $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be an orthogonal basis for S and let $\mathbf{x} \in V$. Show that the best least squares approximation to $\mathbf{x}$ by elements of S is given by

$$\mathbf{p} = \sum_{i=1}^{n} \frac{\langle \mathbf{x}, \mathbf{x}_i \rangle}{\langle \mathbf{x}_i, \mathbf{x}_i \rangle} \mathbf{x}_i$$

36. A (real or complex) scalar u is said to be an nth root of unity if $u^n = 1$.

(a) Show that if u is an nth root of unity and $u \neq 1$, then

$$1 + u + +u^2 + \cdots + u^{n-1} = 0$$

[*Hint*: $1-u^n = (1-u)(1+u+u^2+\cdots+u^{n-1})$]

(b) Let $\omega_n = e^{\frac{2\pi i}{n}}$. Use Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$) to show that ω_n is an nth root of unity.

(c) Show that if j and k are positive integers and if $u = \omega_n^{j-1}$ and $z = \omega_n^{-(k-1)}$, then u, z, and uz are all nth roots of unity.

37. Let ω_n, u and z be defined as in Exercise 36. If F_n is the $n \times n$ Fourier matrix, then its (j, s) entry is

$$f_{js} = \omega_n^{(j-1)(s-1)} = u^{s-1}$$

Let G_n be the matrix defined by

$$g_{sk} = \frac{1}{f_{sk}} = \omega^{-(s-1)(k-1)} = z^{s-1}, \quad \begin{array}{l} 1 \le s \le n, \\ 1 \le k \le n \end{array}$$

Show that the (j, k) entry of $F_n G_n$ is

$$1 + uz + (uz)^2 + \cdots + (uz)^{n-1}$$

38. Use the results from Exercises 36 and 37 to show that F_n is nonsingular and

$$F_n^{-1} = \frac{1}{n} G_n = \frac{1}{n} \overline{F_n}$$

where $\overline{F_n}$ is the matrix whose (i, j) entry is the complex conjugate of f_{ij}.

6 The Gram–Schmidt Orthogonalization Process

In this section, we learn a process for constructing an orthonormal basis for an n-dimensional inner product space V. The method involves using projections to transform an ordinary basis $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ into an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$.