EXAMPLE 7 Given

$$T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

It is easily seen that the eigenvalues of T are $\lambda_1 = 2$ and $\lambda_2 = 3$. If we set $A = S^{-1}TS$, then the eigenvalues of A should be the same as those of T:

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$$

We leave it to the reader to verify that the eigenvalues of this matrix are $\lambda_1 = 2$ and $\lambda_2 = 3$. ■

SECTION 1 EXERCISES

1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ (j) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

(k) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ (l) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

2. Show that the eigenvalues of a triangular matrix are the diagonal elements of the matrix.

3. Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

4. Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

5. Let A and B be $n \times n$ matrices. Show that if none of the eigenvalues of A are equal to 1, then the matrix equation

$$XA + B = X$$

will have a unique solution.

6. Let λ be an eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Use mathematical induction to show that, for $m \geq 1$, λ^m is an eigenvalue of A^m and $\mathbf{x}$ is an eigenvector of A^m belonging to λ^m.

7. Let A be an $n \times n$ matrix and let $B = I - 2A + A^2$.

(a) Show that if $\mathbf{x}$ is an eigenvector of A belonging to an eigenvalue λ of A, then $\mathbf{x}$ is also an eigenvector of B belonging to an eigenvalue μ of B. How are λ and μ related?

(b) Show that if $\lambda = 1$ is an eigenvalue of A, then the matrix B will be singular.

8. An $n \times n$ matrix A is said to be *idempotent* if $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

9. An $n \times n$ matrix is said to be *nilpotent* if $A^k = O$ for some positive integer k. Show that all eigenvalues of a nilpotent matrix are 0.

10. Let A be an $n \times n$ matrix and let $B = A - \alpha I$ for some scalar α. How do the eigenvalues of A and B compare? Explain.

11. Let A be an $n \times n$ matrix and let $B = A + I$. Is it possible for A and B to be similar? Explain.

12. Show that A and A^T have the same eigenvalues. Do they necessarily have the same eigenvectors? Explain.

13. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

will have complex eigenvalues if θ is not a multiple of π. Give a geometric interpretation of this result.

14. Let A be a 2×2 matrix. If $\text{tr}(A) = 8$ and $\det(A) = 12$, what are the eigenvalues of A?

15. Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that

$$\lambda_j = a_{jj} + \sum_{i \neq j}(a_{ii} - \lambda_i) \quad \text{for } j = 1, \ldots, n$$

16. Let A be a 2×2 matrix and let $p(\lambda) = \lambda^2 + b\lambda + c$ be the characteristic polynomial of A. Show that $b = -\operatorname{tr}(A)$ and $c = \det(A)$.

17. Let λ be a nonzero eigenvalue of A and let $\mathbf{x}$ be an eigenvector belonging to λ. Show that $A^m\mathbf{x}$ is also an eigenvector belonging to λ for $m = 1, 2, \ldots$.

18. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. If $A - \lambda I$ has rank k, what is the dimension of the eigenspace corresponding to λ? Explain.

19. Let A be an $n \times n$ matrix. Show that a vector $\mathbf{x}$ in $\mathbb{R}^n$ is an eigenvector belonging to A if and only if the subspace S of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $A\mathbf{x}$ has dimension 1.

20. Let $\alpha = a + bi$ and $\beta = c + di$ be complex scalars and let A and B be matrices with complex entries.

(a) Show that

$$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta} \quad \text{and} \quad \overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$$

(b) Show that the (i, j) entries of $\overline{AB}$ and $\overline{A}\,\overline{B}$ are equal and hence that

$$\overline{AB} = \overline{A}\,\overline{B}$$

21. Let Q be an orthogonal matrix.

(a) Show that if λ is an eigenvalue of Q, then $|\lambda| = 1$.

(b) Show that $|\det(Q)| = 1$.

22. Let Q be an orthogonal matrix with an eigenvalue $\lambda_1 = 1$ and let $\mathbf{x}$ be an eigenvector belonging to λ_1. Show that $\mathbf{x}$ is also an eigenvector of Q^T.

23. Let Q be a 3×3 orthogonal matrix whose determinant is equal to 1.

(a) If the eigenvalues of Q are all real and if they are ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, determine the values of all possible triples of eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$.

(b) In the case that the eigenvalues λ_2 and λ_3 are complex, what are the possible values for λ_1? Explain.

(c) Explain why $\lambda = 1$ must be an eigenvalue of Q.

24. Let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ be eigenvectors of an $n \times n$ matrix A and let S be the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$. Show that S is *invariant* under A (i.e., show that $A\mathbf{x} \in S$ whenever $\mathbf{x} \in S$).

25. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Show that if B is any matrix that commutes with A, then the eigenspace $N(A - \lambda I)$ is invariant under B.

26. Let $B = S^{-1}AS$ and let $\mathbf{x}$ be an eigenvector of B belonging to an eigenvalue λ. Show that $S\mathbf{x}$ is an eigenvector of A belonging to λ.

27. Let A be an $n \times n$ matrix with an eigenvalue λ and let $\mathbf{x}$ be an eigenvector belonging to λ. Let S be a nonsingular $n \times n$ matrix and let α be a scalar. Show that if

$$B = \alpha I - SAS^{-1}, \quad \mathbf{y} = S\mathbf{x}$$

then $\mathbf{y}$ is an eigenvector of B. Determine the eigenvalue of B corresponding to $\mathbf{y}$.

28. Show that if two $n \times n$ matrices A and B have a common eigenvector $\mathbf{x}$ (but not necessarily a common eigenvalue), then $\mathbf{x}$ will also be an eigenvector of any matrix of the form $C = \alpha A + \beta B$.

29. Let A be an $n \times n$ matrix and let λ be a nonzero eigenvalue of A. Show that if $\mathbf{x}$ is an eigenvector belonging to λ, then $\mathbf{x}$ is in the column space of A. Hence, the eigenspace corresponding to λ is a subspace of the column space of A.

30. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ and let A be a linear combination of the rank 1 matrices $\mathbf{u}_1\mathbf{u}_1^T, \mathbf{u}_2\mathbf{u}_2^T, \ldots, \mathbf{u}_n\mathbf{u}_n^T$. If

$$A = c_1\mathbf{u}_1\mathbf{u}_1^T + c_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + c_n\mathbf{u}_n\mathbf{u}_n^T$$

show that A is a symmetric matrix with eigenvalues $c_1, c_2, \ldots, c_n$ and that $\mathbf{u}_i$ is an eigenvector belonging to c_i for each i.

31. Let A be a matrix whose columns all add up to a fixed constant δ. Show that δ is an eigenvalue of A.

32. Let λ_1 and λ_2 be distinct eigenvalues of A. Let $\mathbf{x}$ be an eigenvector of A belonging to λ_1 and let $\mathbf{y}$ be an eigenvector of A^T belonging to λ_2. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

33. Let A and B be $n \times n$ matrices. Show that

(a) If λ is a nonzero eigenvalue of AB, then it is also an eigenvalue of BA.

(b) If $\lambda = 0$ is an eigenvalue of AB, then $\lambda = 0$ is also an eigenvalue of BA.

34. Prove that there do not exist $n \times n$ matrices A and B such that

$$AB - BA = I$$

[*Hint*: See Exercises 10 and 33.]

35. Let $p(\lambda) = (-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda - a_0)$ be a polynomial of degree $n \geq 1$, and let

$$C = \begin{Bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{Bmatrix}$$

(a) Show that if λ_i is a root of $p(\lambda) = 0$, then λ_i is an eigenvalue of C with eigenvector $\mathbf{x} = (\lambda_i^{n-1}, \lambda_i^{n-2}, \ldots, \lambda_i, 1)^T$.

(b) Use part (a) to show that if $p(\lambda)$ has n distinct roots, then $p(\lambda)$ is the characteristic polynomial of C.

The matrix C is called the *companion matrix* of $p(\lambda)$.

36. The result given in Exercise 35(b) holds even if all the eigenvalues of $p(\lambda)$ are not distinct. Prove this as follows:

(a) Let

$$D_m(\lambda) = \begin{bmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 \\ 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & -\lambda \end{bmatrix}$$

and use mathematical induction to prove that

$$\det(D_m(\lambda)) = (-1)^m(a_m\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0)$$

(b) Show that

$$\begin{aligned} \det(C - \lambda I) &= (a_{n-1} - \lambda)(-\lambda)^{n-1} - \det(D_{n-2}) \\ &= p(\lambda) \end{aligned}$$

2 Systems of Linear Differential Equations

Eigenvalues play an important role in the solution of systems of linear differential equations. In this section, we see how they are used in the solution of systems of linear differential equations with constant coefficients. We begin by considering systems of first-order equations of the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

where $y_i = f_i(t)$ is a function in $C^1[a, b]$ for each i. If we let

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad \mathbf{Y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

then the system can be written in the form

$$\mathbf{Y}' = A\mathbf{Y}$$

$\mathbf{Y}$ and $\mathbf{Y}'$ are both vector functions of t. Let us consider the simplest case first. When $n = 1$, the system is simply

$$y' = ay \tag{1}$$

Clearly, any function of the form

$$y(t) = ce^{at} \qquad (c \text{ an arbitrary constant})$$