SECTION 3 EXERCISES

1. In each of the following, factor the matrix A into a product XDX^{-1}, where D is diagonal:

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$

2. For each of the matrices in Exercise 1, use the XDX^{-1} factorization to compute A^6.

3. For each of the nonsingular matrices in Exercise 1, use the XDX^{-1} factorization to compute A^{-1}.

4. For each of the following, find a matrix B such that $B^2 = A$:

(a) $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

5. Let A be a nondefective $n \times n$ matrix with diagonalizing matrix X. Show that the matrix $Y = (X^{-1})^T$ diagonalizes A^T.

6. Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

7. Show that any 3×3 matrix of the form

$$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$$

is defective.

8. For each of the following, find all possible values of the scalar α that make the matrix defective or show that no such values exist:

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & \alpha \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & \alpha \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 6 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & \alpha \end{bmatrix}$

(e) $\begin{bmatrix} 3\alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$

(f) $\begin{bmatrix} 3\alpha & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}$

(g) $\begin{bmatrix} \alpha+2 & 1 & 0 \\ 0 & \alpha+2 & 0 \\ 0 & 0 & 2\alpha \end{bmatrix}$

(h) $\begin{bmatrix} \alpha+2 & 0 & 0 \\ 0 & \alpha+2 & 1 \\ 0 & 0 & 2\alpha \end{bmatrix}$

9. Let A be a 4×4 matrix and let λ be an eigenvalue of multiplicity 3. If $A - \lambda I$ has rank 1, is A defective? Explain.

10. Let A be an $n \times n$ matrix with positive real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let $\mathbf{x}_i$ be an eigenvector belonging to λ_i for each i, and let $\mathbf{x} = \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$.

(a) Show that $A^m\mathbf{x} = \sum_{i=1}^{n} \alpha_i\lambda_i^m\mathbf{x}_i$.

(b) Show that if $\lambda_1 = 1$, then $\lim_{m\to\infty} A^m\mathbf{x} = \alpha_1\mathbf{x}_1$.

11. Let A be a $n \times n$ matrix with real entries and let $\lambda_1 = a + bi$ (where a and b are real and $b \neq 0$) be an eigenvalue of A. Let $\mathbf{z}_1 = \mathbf{x} + i\,\mathbf{y}$ (where $\mathbf{x}$ and $\mathbf{y}$ both have real entries) be an eigenvector belonging to λ_1 and let $\mathbf{z}_2 = \mathbf{x} - i\,\mathbf{y}$.

(a) Explain why $\mathbf{z}_1$ and $\mathbf{z}_2$ must be linearly independent.

(b) Show that $\mathbf{y} \neq \mathbf{0}$ and that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

12. Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity n. Show that A is diagonalizable if and only if $A = \lambda I$.

13. Show that a nonzero nilpotent matrix is defective.

14. Let A be a diagonalizable matrix and let X be the diagonalizing matrix. Show that the column vectors of X that correspond to nonzero eigenvalues of A form a basis for $R(A)$.

15. It follows from Exercise 14 that, for a diagonalizable matrix, the number of nonzero eigenvalues (counted according to multiplicity) equals the rank of the matrix. Give an example of a defective matrix whose rank is not equal to the number of nonzero eigenvalues.

16. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A whose eigenspace has dimension k, where $1 < k < n$. Any basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for the eigenspace can be extended to a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for $\mathbb{R}^n$. Let $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $B = X^{-1}AX$.

(a) Show that B is of the form

$$\begin{Bmatrix} \lambda I & B_{12} \\ O & B_{22} \end{Bmatrix}$$

where I is the $k \times k$ identity matrix.

(b) Use Theorem 1.1 to show that λ is an eigenvalue of A with multiplicity at least k.

17. Let $\mathbf{x}$, $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$, $n \geq 2$, and let $A = \mathbf{x}\mathbf{y}^T$. Show that

(a) $\lambda = 0$ is an eigenvalue of A with $n - 1$ linearly independent eigenvectors and consequently has multiplicity at least $n-1$ (see Exercise 16).

(b) the remaining eigenvalue of A is

$$\lambda_n = \operatorname{tr} A = \mathbf{x}^T\mathbf{y}$$

and $\mathbf{x}$ is an eigenvector belonging to λ_n.

(c) if $\lambda_n = \mathbf{x}^T\mathbf{y} \neq 0$, then A is diagonalizable.

18. Let A be a diagonalizable $n \times n$ matrix. Prove that if B is any matrix that is similar to A, then B is diagonalizable.

19. Show that if A and B are two $n \times n$ matrices with the same diagonalizing matrix X, then $AB = BA$.

20. Let T be an upper triangular matrix with distinct diagonal entries (i.e., $t_{ii} \neq t_{jj}$ whenever $i \neq j$). Show that there is an upper triangular matrix R that diagonalizes T.

21. Each year, employees at a company are given the option of donating to a local charity as part of a payroll deduction plan. In general, 80 percent of the employees enrolled in the plan in any one year will choose to sign up again the following year, and 30 percent of the unenrolled will choose to enroll the following year. Determine the transition matrix for the Markov process and find the steady-state vector. What percentage of employees would you expect to find enrolled in the program in the long run?

22. The city of Mawtookit maintains a constant population of 300,000 people from year to year. A political science study estimated that there were 150,000 Independents, 90,000 Democrats, and 60,000 Republicans in the town. It was also estimated that each year 20 percent of the Independents become Democrats and 10 percent become Republicans. Similarly, 20 percent of the Democrats become Independents and 10 percent become Republicans, while 10 percent of the Republicans defect to the Democrats and 10 percent become Independents each year. Let

$$\mathbf{x} = \begin{Bmatrix} 150{,}000 \\ 90{,}000 \\ 60{,}000 \end{Bmatrix}$$

and let $\mathbf{x}^{(1)}$ be a vector representing the number of people in each group after one year.

(a) Find a matrix A such that $A\mathbf{x} = \mathbf{x}^{(1)}$.

(b) Show that $\lambda_1 = 1.0$, $\lambda_2 = 0.5$, and $\lambda_3 = 0.7$ are the eigenvalues of A, and factor A into a product XDX^{-1}, where D is diagonal.

(c) Which group will dominate in the long run? Justify your answer by computing $\lim_{n\to\infty} A^n\mathbf{x}$.

23. Let

$$A = \begin{Bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \end{Bmatrix}$$

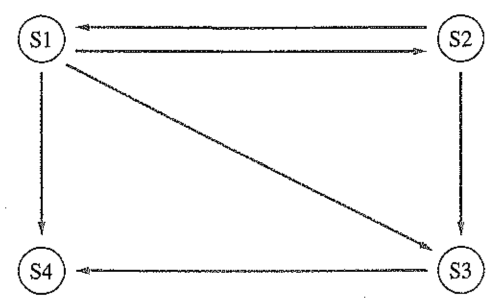
be a transition matrix for a Markov process.

(a) Compute $\det(A)$ and $\text{trace}(A)$ and make use of those values to determine the eigenvalues of A.

(b) Explain why the Markov process must converge to a steady-state vector.

(c) Show that $\mathbf{y} = (16, 15, 15)^T$ is an eigenvector of A. How is the steady-state vector related to $\mathbf{y}$?

24. Consider a Web network consisting of only four sites that are linked together as shown in the accompanying diagram. If the Google PageRank algorithm is used to rank these pages, determine the transition matrix A. Assume that the Web surfer will follow a link on the current page 85 percent of the time.



25. Let A be an $n \times n$ stochastic matrix and let $\mathbf{e}$ be the vector in $\mathbb{R}^n$ whose entries are all equal to 1. Show that $\mathbf{e}$ is an eigenvector of A^T. Explain why a stochastic matrix must have $\lambda = 1$ as an eigenvalue.

26. The transition matrix in Example 5 has the property that both its rows and its columns add up to 1. In general, a matrix A is said to be *doubly stochastic* if both A and A^T are stochastic. Let A be an $n \times n$ doubly stochastic matrix whose eigenvalues satisfy

$$\lambda_1 = 1 \quad \text{and} \quad |\lambda_j| < 1 \quad \text{for } j = 2, 3, \ldots, n$$

Show that if $\mathbf{e}$ is the vector in $\mathbb{R}^n$ whose entries are all equal to 1, then the Markov chain will converge to the steady-state vector $\mathbf{x} = \frac{1}{n}\mathbf{e}$ for any starting vector $\mathbf{x}_0$. Thus, for a doubly stochastic transition matrix, the steady-state vector will assign equal probabilities to all possible outcomes.

27. Let A be the PageRank transition matrix and let $\mathbf{x}_k$ be a vector in the Markov chain with starting probability vector $\mathbf{x}_0$. Since n is very large, the direct multiplication $\mathbf{x}_{k+1} = A\mathbf{x}_k$ is computationally intensive. However, the computation can be simplified dramatically if we take advantage of the structured components of A given in equation (5). Because M is sparse, the multiplication $\mathbf{w}_k = M\mathbf{x}_k$ is computationally much simpler. Show that if we set

$$\mathbf{b} = \frac{1-p}{n}\mathbf{e}$$

then

$$E\mathbf{x}_k = \mathbf{e} \quad \text{and} \quad \mathbf{x}_{k+1} = p\mathbf{w}_k + \mathbf{b}$$

where M, E, $\mathbf{e}$, and p are as defined in equation (5).

28. Use the definition of the matrix exponential to compute e^A for each of the following matrices:

(a) $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

29. Compute e^A for each of the following matrices:

(a) $A = \begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix}$ (b) $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

30. In each of the following, solve the initial value problem $\mathbf{Y}' = A\mathbf{Y}$, $\mathbf{Y}(0) = \mathbf{Y}_0$, by computing $e^{tA}\mathbf{Y}_0$:

(a) $A = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$, $\mathbf{Y}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$, $\mathbf{Y}_0 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, $\mathbf{Y}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$, $\mathbf{Y}_0 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

31. Let λ be an eigenvalue of an $n \times n$ matrix A and let $\mathbf{x}$ be an eigenvector belonging to λ. Show that e^λ is an eigenvalue of e^A and $\mathbf{x}$ is an eigenvector of e^A belonging to e^λ.

32. Show that e^A is nonsingular for any diagonalizable matrix A.

33. Let A be a diagonalizable matrix with characteristic polynomial

$$p(\lambda) = a_1\lambda^n + a_2\lambda^{n-1} + \cdots + a_{n+1}$$

(a) Show that if D is a diagonal matrix whose diagonal entries are the eigenvalues of A, then

$$p(D) = a_1 D^n + a_2 D^{n-1} + \cdots + a_{n+1} I = O$$

(b) Show that $p(A) = O$.

(c) Show that if $a_{n+1} \neq 0$, then A is nonsingular and $A^{-1} = q(A)$ for some polynomial q of degree less than n.

4 Hermitian Matrices

Let $\mathbb{C}^n$ denote the vector space of all n-tuples of complex numbers. The set $\mathbb{C}$ of all complex numbers will be taken as our field of scalars. We have already seen that a matrix A with real entries may have complex eigenvalues and eigenvectors. In this section, we study matrices with complex entries and look at the complex analogues of symmetric and orthogonal matrices.