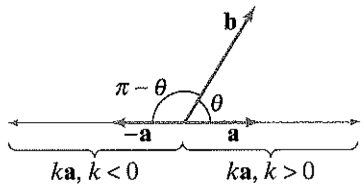

Figure 65 If the angle between **a** and **b** is θ, then the angle between $k\mathbf{a}$ and **b** is either θ (if $k > 0$) or $\pi - \theta$ (if $k < 0$).

If $k < 0$, then the direction of $(k\mathbf{a}) \times \mathbf{b}$ is the same as that of $(-\mathbf{a}) \times \mathbf{b}$, which is seen to be the same as that of $-(\mathbf{a} \times \mathbf{b})$ and thus the same as that of $k(\mathbf{a} \times \mathbf{b})$. The angle between $k\mathbf{a}$ and **b** is therefore $\pi - \theta$, where θ is the angle between **a** and **b**. (See Figure 65.) Thus,

$$\|(k\mathbf{a}) \times \mathbf{b}\| = \|k\mathbf{a}\| \, \|\mathbf{b}\| \sin(\pi - \theta) = |k| \, \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta = \|k(\mathbf{a} \times \mathbf{b})\|.$$

So, again, it follows that $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b})$. ■

4 Exercises

Evaluate the determinants in Exercises 1–4.

1. $\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 5 \\ -1 & 6 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ -1 & 0 & 3 \end{vmatrix}$

4. $\begin{vmatrix} -2 & 0 & \frac{1}{2} \\ 3 & 6 & -1 \\ 4 & -8 & 2 \end{vmatrix}$

In Exercises 5–7, calculate the indicated cross products, using both formulas (2) and (3).

5. $(1, 3, -2) \times (-1, 5, 7)$

6. $(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$

7. $(\mathbf{i} + \mathbf{j}) \times (-3\mathbf{i} + 2\mathbf{j})$

8. Prove property 3 of cross products, using properties 1 and 2.

9. If $\mathbf{a} \times \mathbf{b} = 3\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$, what is $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$?

10. Calculate the area of the parallelogram having vertices $(1, 1)$, $(3, 2)$, $(1, 3)$, and $(-1, 2)$.

11. Calculate the area of the parallelogram having vertices $(1, 2, 3)$, $(4, -2, 1)$, $(-3, 1, 0)$, and $(0, -3, -2)$.

12. Find a unit vector that is perpendicular to both $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{i} + \mathbf{k}$.

13. If $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$, what can you say about the geometric relation between **a**, **b**, and **c**?

Compute the area of the triangles described in Exercises 14–17.

14. The triangle determined by the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$

15. The triangle determined by the vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

16. The triangle having vertices $(1, 1)$, $(-1, 2)$, and $(-2, -1)$

17. The triangle having vertices $(1, 0, 1)$, $(0, 2, 3)$, and $(-1, 5, -2)$

18. Find the volume of the parallelepiped determined by $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{k}$, and $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

19. What is the volume of the parallelepiped with vertices $(3, 0, -1)$, $(4, 2, -1)$, $(-1, 1, 0)$, $(3, 1, 5)$, $(0, 3, 0)$, $(4, 3, 5)$, $(-1, 2, 6)$, and $(0, 4, 6)$?

20. Verify that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

21. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ using Exercise 20.

22. Use geometry to show that $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})|$.

23. (a) Show that the area of the triangle with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ is given by the absolute value of the expression

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

(b) Use part (a) to find the area of the triangle with vertices $(1, 2)$, $(2, 3)$, and $(-4, -4)$.

24. Suppose that **a**, **b**, and **c** are noncoplanar vectors in $\mathbf{R}^3$, so that they determine a tetrahedron as in Figure 66.

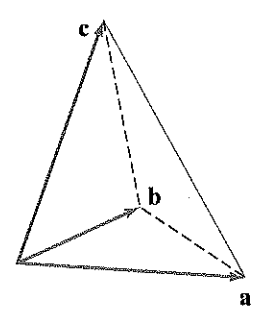

Figure 66 The tetrahedron of Exercise 24.

Give a formula for the surface area of the tetrahedron in terms of **a**, **b**, and **c**. (Note: More than one formula is possible.)