PROOF We focus on the total area enclosed by the elliptical orbit. The area of an ellipse whose semimajor and semiminor axes have lengths a and b, respectively, is πab. This area must also be that swept by the planet in the time interval $[0, T]$. Thus, we have

$$\begin{aligned} \pi ab &= \int_0^T \frac{dA}{dt}\,dt \\ &= \int_0^T \frac{1}{2}c\,dt \qquad \text{by equation (18)} \\ &= \frac{1}{2}cT. \end{aligned}$$

Hence,

$$T = \frac{2\pi ab}{c}, \quad \text{so} \quad T^2 = \frac{4\pi^2 a^2 b^2}{c^2}. \tag{19}$$

Now, b and c are related to a, so these quantities must be replaced before we are done. In particular, from equation (16), $b^2 = p^2/(1 - e^2)$, so

$$b^2 = pa.$$

Also

$$p = \frac{c^2}{GM}.$$

(See equations (12) and (13).) With these substitutions, the result in (19) becomes

$$T^2 = \frac{4\pi^2 a^2 (pa)}{pGM} = \left(\frac{4\pi^2}{GM}\right) a^3.$$

This last equation shows that T^2 is proportional to a^3, but it says even more: The constant of proportionality $4\pi^2/GM$ depends entirely on the mass of the sun—the constant is the same for *any* planet that might revolve around the sun. ■

1 Exercises

In Exercises 1–6, sketch the images of the following paths, using arrows to indicate the direction in which the parameter increases:

1. $\begin{cases} x = 2t - 1 \\ y = 3 - t \end{cases}, \quad -1 \le t \le 1$

2. $\mathbf{x}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$

3. $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}, \quad -6\pi \le t \le 6\pi$

4. $\begin{cases} x = 3\cos t \\ y = 2\sin 2t \end{cases}, \quad 0 \le t \le 2\pi$

5. $\mathbf{x}(t) = (t, 3t^2 + 1, 0)$

6. $\mathbf{x}(t) = (t, t^2, t^3)$

Calculate the velocity, speed, and acceleration of the paths given in Exercises 7–10.

7. $\mathbf{x}(t) = (3t - 5)\mathbf{i} + (2t + 7)\mathbf{j}$

8. $\mathbf{x}(t) = 5\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$

9. $\mathbf{x}(t) = (t\sin t, t\cos t, t^2)$

10. $\mathbf{x}(t) = (e^t, e^{2t}, 2e^t)$

In Exercises 11–14, (a) use a computer to give a plot of the given path $\mathbf{x}$ *over the indicated interval for* t*; identify the direction in which* t *increases. (b) Show that the path lies on the given surface* S.

11. $\mathbf{x}(t) = (3\cos\pi t, 4\sin\pi t, 2t)$, $-4 \le t \le 4$; S is elliptical cylinder $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

12. $\mathbf{x}(t) = (t\cos t, t\sin t, t)$, $-20 \le t \le 20$; S is cone $z^2 = x^2 + y^2$.

13. $\mathbf{x}(t) = (t\sin 2t, t\cos 2t, t^2)$, $-6 \le t \le 6$; S is paraboloid $z = x^2 + y^2$.

14. $\mathbf{x}(t) = (2\cos t, 2\sin t, 3\sin 8t)$, $0 \le t \le 2\pi$; S is cylinder $x^2 + y^2 = 4$.

In Exercises 15–18, find an equation for the line tangent to the given path at the indicated value for the parameter.

15. $\mathbf{x}(t) = te^{-t}\,\mathbf{i} + e^{3t}\,\mathbf{j}$, $t = 0$

16. $\mathbf{x}(t) = 4\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j} + 5t\,\mathbf{k}$, $t = \pi/3$

17. $\mathbf{x}(t) = (t^2, t^3, t^5)$, $t = 2$

18. $\mathbf{x}(t) = (\cos(e^t), 3 - t^2, t)$, $t = 1$

19. (a) Sketch the path $\mathbf{x}(t) = (t, t^3 - 2t + 1)$.

(b) Calculate the line tangent to $\mathbf{x}$ when $t = 2$.

(c) Describe the image of $\mathbf{x}$ by an equation of the form $y = f(x)$ by eliminating t.

(d) Verify your answer in part (b) by recalculating the tangent line, using your result in part (c).

Exercises 20–23 concern Roger Ramjet and his trajectory when he is shot from a cannon as in Example 6 of this section.

20. Verify that Roger Ramjet's path in Example 6 is indeed a parabola.

21. Suppose that Roger is fired from the cannon with an angle of inclination θ of 60° and an initial speed v_0 of 100 ft/sec. What is the maximum height Roger attains?

22. Suppose that Roger is fired from the cannon with an angle of inclination θ of 60° and that he hits the ground 1/2 mile from the cannon. What, then, was Roger's initial speed?

23. If Roger is fired from the cannon with an initial speed of 250 ft/sec, what angle of inclination θ should be used so that Roger hits the ground 1500 ft from the cannon?

24. Gertrude is aiming a Super Drencher water pistol at Egbert, who is 1.6 m tall and is standing 5 m away. Gertrude holds the water gun 1 m above ground at an angle α of elevation. (See Figure 14.)

(a) If the water pistol fires with an initial speed of 7 m/sec and an elevation angle of 45°, does Egbert get wet?

(b) If the water pistol fires with an initial speed of 8 m/sec, what possible angles of elevation will cause Egbert to get wet? (Note: You will want to use a computer algebra system or a graphics calculator for this part.)

25. A malfunctioning rocket is traveling according to the path $\mathbf{x}(t) = \left(e^{2t}, 3t^3 - 2t, t - \frac{1}{t}\right)$ in the hope of reaching a repair station at the point $(7e^4, 35, 5)$. (Here t represents time in minutes and spatial coordinates are measured in miles.) At $t = 2$, the rocket's engines suddenly cease. Will the rocket coast into the repair station?

26. Two billiard balls are moving on a (coordinatized) pool table according to the respective paths $\mathbf{x}(t) = \left(t^2 - 2, \frac{t^2}{2} - 1\right)$ and $\mathbf{y}(t) = (t, 5 - t^2)$, where t represents time measured in seconds.

(a) When and where do the balls collide?

(b) What is the angle formed by the paths of the balls at the collision point?

27. Establish part 1 of Proposition 1.4 in this section: If $\mathbf{x}$ and $\mathbf{y}$ are differentiable paths in $\mathbf{R}^n$, show that

$$\frac{d}{dt}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{y}\cdot\frac{d\mathbf{x}}{dt} + \mathbf{x}\cdot\frac{d\mathbf{y}}{dt}.$$

28. Establish part 2 of Proposition 1.4 in this section: If $\mathbf{x}$ and $\mathbf{y}$ are differentiable paths in $\mathbf{R}^3$, show that

$$\frac{d}{dt}(\mathbf{x}\times\mathbf{y}) = \frac{d\mathbf{x}}{dt}\times\mathbf{y} + \mathbf{x}\times\frac{d\mathbf{y}}{dt}.$$

29. Prove Proposition 1.7.

30. (a) Show that the path $\mathbf{x}(t) = (\cos t, \cos t \sin t, \sin^2 t)$ lies on a unit sphere.

(b) Verify that $\mathbf{x}(t)$ is always perpendicular to the velocity vector $\mathbf{v}(t)$.

(c) Use Proposition 1.7 to show that if a differentiable path lies on a sphere centered at the origin, then its position vector is always perpendicular to its velocity vector.

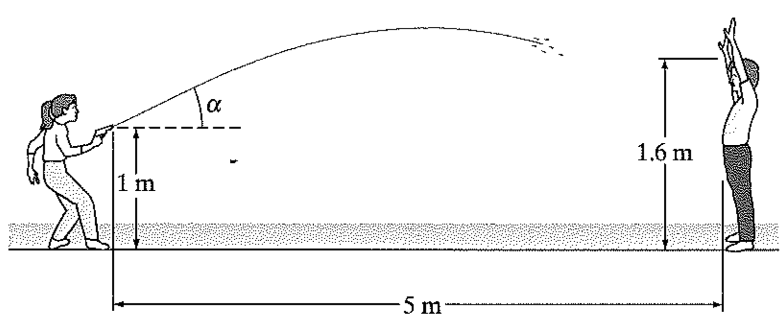

Figure 14 Figure for Exercise 24.