Moreover, we may differentiate the equation $\mathbf{N}\cdot\mathbf{N}=1$ to find

$$b(s)=\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}=-\mathbf{N}\cdot\frac{d\mathbf{N}}{ds},$$

which implies that $b(s)$ is zero. Hence, equation (19) becomes

$$\frac{d\mathbf{N}}{ds}=-\kappa\mathbf{T}+\tau\mathbf{B}.$$

The formulas for $d\mathbf{T}/ds$, $d\mathbf{N}/ds$, and $d\mathbf{B}/ds$ are usually taken together as

$$\begin{cases}\mathbf{T}'(s)=\kappa\mathbf{N}\\ \mathbf{N}'(s)=-\kappa\mathbf{T}+\tau\mathbf{B}\\ \mathbf{B}'(s)=-\tau\mathbf{N}\end{cases}$$

and are known as the **Frenet–Serret formulas** for a curve in space. They are so named for Frédéric-Jean Frenet and Joseph Alfred Serret, who published them separately in 1852 and 1851, respectively. The Frenet–Serret formulas give a system of differential equations for a curve and are key to proving a result like Theorem 2.5. They are often written in matrix form, in which case, they have an especially appealing appearance, namely,

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\ -\kappa&0&\tau\\ 0&-\tau&0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}.$$

2 Exercises

Calculate the length of each of the paths given in Exercises 1–6.

1. $\mathbf{x}(t)=(2t+1,7-3t)$, $-1\le t\le 2$
2. $\mathbf{x}(t)=t^2\,\mathbf{i}+\frac{2}{3}(2t+1)^{3/2}\,\mathbf{j}$, $0\le t\le 4$
3. $\mathbf{x}(t)=(\cos 3t,\sin 3t,2t^{3/2})$, $0\le t\le 2$
4. $\mathbf{x}(t)=7\mathbf{i}+t\,\mathbf{j}+t^2\mathbf{k}$, $1\le t\le 3$
5. $\mathbf{x}(t)=(t^3,3t^2,6t)$, $-1\le t\le 2$
6. $\mathbf{x}(t)=(\ln(\cos t),\cos t,\sin t)$, $\frac{\pi}{6}\le t\le\frac{\pi}{3}$
7. $\mathbf{x}(t)=(\ln t,t^2/2,\sqrt{2}t)$, $1\le t\le 4$
8. $\mathbf{x}(t)=(2t\cos t,2t\sin t,2\sqrt{2}t^2)$, $0\le t\le 3$
9. The path $\mathbf{x}(t)=(a\cos^3 t,a\sin^3 t)$, where a is a positive constant, traces a curve known as an **astroid** or a **hypocycloid of four cusps.** Sketch this curve and find its total length. (Be careful when you do this.)
10. If f is a continuously differentiable function, show how Definition 2.1 may be used to establish the formula

$$L=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

for the length of the curve $y=f(x)$ between $(a,f(a))$ and $(b,f(b))$.

11. Use Exercise 10 or Definition 2.1 (or both) to calculate the length of the line segment $y=mx+b$ between (x_0,y_0) and (x_1,y_1). Explain your result with an appropriate sketch.
12. (a) Calculate the length of the line segment determined by the path

$$\mathbf{x}(t)=(a_1t+b_1,a_2t+b_2)$$

as t varies from t_0 to t_1.

(b) Compare your result with that of Exercise 11.

(c) Now calculate the length of the line segment determined by the path $\mathbf{x}(t)=\mathbf{a}\,t+\mathbf{b}$ as t varies from t_0 to t_1.

13. This problem concerns the path $\mathbf{x}=|t-1|\,\mathbf{i}+|t|\,\mathbf{j}$, $-2\le t\le 2$.
    (a) Sketch this path.
    (b) The path fails to be of class C^1 but is piecewise C^1. Explain.
    (c) Calculate the length of the path.
14. Consider the path $\mathbf{x}(t)=(e^{-t}\cos t,e^{-t}\sin t)$.