If we write the component functions of $\mathbf{X}$ as

$$\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t)),$$

we find that

$$\mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix}$$

$$= \left(\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}\right)\mathbf{i} + \left(\frac{\partial x}{\partial t}\frac{\partial z}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial z}{\partial t}\right)\mathbf{j} + \left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right)\mathbf{k}.$$

Using the notation of the Jacobian, we obtain

$$\mathbf{N}(s,t) = \mathbf{T}_s \times \mathbf{T}_t = \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k}. \quad (7)$$

This alternative formula for the normal vector to a smooth parametrized surface will prove useful to us on occasion. For the moment, we take its magnitude:

$$\|\mathbf{N}(s,t)\| = \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2}.$$

Hence, formula (6) may also be written as

Surface area of S

$$= \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2}\, ds\, dt. \quad (8)$$

EXAMPLE 12 Find the surface area of the torus described in Example 5. Recall that the torus is parametrized as

$$\begin{cases} x = (a + b\cos t)\cos s \\ y = (a + b\cos t)\sin s \qquad 0 \le s, t \le 2\pi, \quad a > b > 0. \\ z = b\sin t \end{cases}$$

Thus,

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} -(a + b\cos t)\sin s & -b\sin t\, \cos s \\ (a + b\cos t)\cos s & -b\sin t\, \sin s \end{vmatrix}$$

$$= (a + b\cos t)(b\sin t\sin^2 s + b\sin t\, \cos^2 s)$$

$$= (a + b\cos t)b\sin t,$$

$$\frac{\partial(x,z)}{\partial(s,t)} = \begin{vmatrix} -(a + b\cos t)\sin s & -b\sin t\, \cos s \\ 0 & b\cos t \end{vmatrix}$$

$$= -(a + b\cos t)b\cos t\, \sin s,$$

and

$$\frac{\partial(y,z)}{\partial(s,t)} = \begin{vmatrix} (a + b\cos t)\cos s & -b\sin t\, \sin s \\ 0 & b\cos t \end{vmatrix}$$

$$= (a + b\cos t)b\cos t\, \cos s.$$

By formula (8), we have

Surface area

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{(a + b\cos t)^2[b^2\sin^2 t + b^2\cos^2 t\sin^2 s + b^2\cos^2 t\cos^2 s]}\, ds\, dt.$$

Using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ twice, we simplify the integral to

$$\int_0^{2\pi}\int_0^{2\pi} (a + b\cos t)b\, ds\, dt = \int_0^{2\pi} 2\pi b(a + b\cos t)\, dt$$

$$= 2\pi b(at + b\sin t)\Big|_0^{2\pi}$$

$$= 4\pi^2 ab.$$

◆

EXAMPLE 13 Suppose that a smooth surface is described as the graph of a C^1 function $f(x,y)$, that is, by the equation $z = f(x,y)$, where (x,y) varies through a plane region D. Then the standard parametrization $\mathbf{X}(s,t) = (s, t, f(s,t))$ implies

$$\mathbf{T}_s \times \mathbf{T}_t = -f_s\,\mathbf{i} - f_t\,\mathbf{j} + \mathbf{k}.$$

(See Example 9.) Formula (6) yields

$$\text{Surface area} = \iint_D \|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt = \iint_D \sqrt{f_s^2 + f_t^2 + 1}\, ds\, dt.$$

Since $x = s$, $y = t$ in this parametrization of the graph, we conclude that

Surface area of the graph of $f(x,y)$ over D

$$= \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy. \quad (9)$$

◆

One final note: It is not at all clear that either formula (6) or formula (8) depends only on the underlying surface $S = \mathbf{X}(D)$ and not on the particular parametrization $\mathbf{X}$. These formulas are independent of the parametrization, as we shall observe in the following section, in the context of general surface integrals.

1 Exercises

1. Let $\mathbf{X}: \mathbf{R}^2 \to \mathbf{R}^3$ be the parametrized surface given by

$$\mathbf{X}(s,t) = (s^2 - t^2, s + t, s^2 + 3t).$$

(a) Determine a normal vector to this surface at the point

$$(3,1,1) = \mathbf{X}(2,-1).$$

(b) Find an equation for the plane tangent to this surface at the point $(3, 1, 1)$.

2. Find an equation for the plane tangent to the torus $\mathbf{X}(s,t) = ((5 + 2\cos t)\cos s, (5 + 2\cos t)\sin s, 2\sin t)$ at the point $\left((5 - \sqrt{3})/\sqrt{2}, (5 - \sqrt{3})/\sqrt{2}, 1\right)$.

3. Find an equation for the plane tangent to the surface

$$x = e^s, \qquad y = t^2e^{2s}, \qquad z = 2e^{-s} + t$$

at the point $(1, 4, 0)$.

4. Let $\mathbf{X}(s, t) = (s^2 \cos t, s^2 \sin t, s)$, $-3 \le s \le 3$, $0 \le t \le 2\pi$.

(a) Find a normal vector at $(s, t) = (-1, 0)$.

(b) Determine the tangent plane at the point $(1, 0, -1)$.

(c) Find an equation for the image of $\mathbf{X}$ in the form $F(x, y, z) = 0$.

5. Consider the parametrized surface $\mathbf{X}(s, t) = (s, s^2 + t, t^2)$.

(a) Graph this surface for $-2 \le s \le 2$, $-2 \le t \le 2$. (Using a computer may help.)

(b) Is the surface smooth?

(c) Find an equation for the tangent plane at the point $(1, 0, 1)$.

6. Describe the parametrized surface of Exercise 1 by an equation of the form $z = f(x, y)$.

7. Let S be the surface parametrized by $x = s \cos t$, $y = s \sin t$, $z = s^2$, where $s \ge 0$, $0 \le t \le 2\pi$.

(a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$.

(b) Sketch the graph of S. Can you recognize S as a familiar surface?

(c) Describe S by an equation of the form $z = f(x, y)$.

(d) Using your answer in part (c), discuss whether S has a tangent plane at every point.

8. Verify that the image of the parametrized surface

$$\mathbf{X}(s, t) = (2 \sin s \ \cos t, 3 \sin s \ \sin t, \cos s),$$
$$0 \le s \le \pi, \quad 0 \le t \le 2\pi,$$

is an ellipsoid.

9. Verify that, for the torus of Example 5, the s-coordinate curve, when $t = t_0$, is a circle of radius $a + b \cos t_0$.

10. The surface in $\mathbf{R}^3$ parametrized by

$$\mathbf{X}(r, \theta) = (r \cos\theta, r \sin\theta, \theta), \quad r \ge 0, \quad -\infty < \theta < \infty,$$

is called a **helicoid**.

(a) Describe the r-coordinate curve when $\theta = \pi/3$. Give a general description of the r-coordinate curves.

(b) Describe the θ-coordinate curve when $r = 1$. Give a general description of the θ-coordinate curves.

(c) Sketch the graph of the helicoid (perhaps using a computer) for $0 \le r \le 1$, $0 \le \theta \le 4\pi$. Can you see why the surface is called a helicoid?

11. Given the sphere of radius 2 centered at $(2, -1, 0)$, find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

(a) by considering the sphere as the graph of the function

$$f(x, y) = \sqrt{4 - (x - 2)^2 - (y + 1)^2};$$

(b) by considering the sphere as a level surface of the function

$$F(x, y, z) = (x - 2)^2 + (y + 1)^2 + z^2;$$

(c) by considering the sphere as the surface parametrized by

$$\mathbf{X}(s, t) = (2 \sin s \ \cos t + 2, 2 \sin s \ \sin t - 1, 2 \cos s).$$

In Exercises 12–15, represent the given surface as a piecewise smooth parametrized surface.

12. The lower hemisphere $x^2 + y^2 + z^2 = 9$, including the equatorial circle.

13. The part of the cylinder $x^2 + z^2 = 4$ lying between $y = -1$ and $y = 3$.

14. The closed triangular region in $\mathbf{R}^3$ with vertices $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 5)$.

15. The hyperboloid $z^2 - x^2 - y^2 = 1$. (Hint: Use two maps to parametrize the surface.)

16. This problem concerns the parametrized surface $\mathbf{X}(s, t) = (s^3, t^3, st)$.

(a) Find an equation of the plane tangent to this surface at the point $(1, -1, -1)$.

(b) Is this surface smooth? Why or why not?

(c) Use a computer to graph this surface for $-1 \le s \le 1$, $-1 \le t \le 1$.

(d) Verify that this surface may also be described by the xyz-coordinate equation $z = \sqrt[3]{xy}$. Try using a computer to graph the surface when described in this form. Many software systems will have trouble, or will provide an incomplete graph, which is one reason why parametric descriptions of surfaces are desirable.

17. The surface given parametrically by $\mathbf{X}(s, t) = (st, t, s^2)$ is known as the **Whitney umbrella**.

(a) Verify that this surface may also be described by the xyz-coordinate equation $y^2z = x^2$.

(b) Is $\mathbf{X}$ smooth?

(c) Use a computer to graph this surface for $-2 \le s \le 2$, $-2 \le t \le 2$.

(d) Some points (x, y, z) of the surface do not correspond to a single parameter point (s, t). Which ones? Explain how this relates to the graph.

(e) Give an equation of the plane tangent to this surface at the point $(2, 1, 4)$.

3. Find an equation for the plane tangent to the surface

$$x = e^s, \qquad y = t^2 e^{2s}, \qquad z = 2e^{-s} + t$$

at the point $(1, 4, 0)$.

4. Let $\mathbf{X}(s, t) = (s^2 \cos t, s^2 \sin t, s)$, $-3 \le s \le 3$, $0 \le t \le 2\pi$.

(a) Find a normal vector at $(s, t) = (-1, 0)$.

(b) Determine the tangent plane at the point $(1, 0, -1)$.

(c) Find an equation for the image of $\mathbf{X}$ in the form $F(x, y, z) = 0$.

5. Consider the parametrized surface $\mathbf{X}(s, t) = (s, s^2 + t, t^2)$.

(a) Graph this surface for $-2 \le s \le 2$, $-2 \le t \le 2$. (Using a computer may help.)

(b) Is the surface smooth?

(c) Find an equation for the tangent plane at the point $(1, 0, 1)$.

6. Describe the parametrized surface of Exercise 1 by an equation of the form $z = f(x, y)$.

7. Let S be the surface parametrized by $x = s \cos t$, $y = s \sin t$, $z = s^2$, where $s \ge 0$, $0 \le t \le 2\pi$.

(a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$.

(b) Sketch the graph of S. Can you recognize S as a familiar surface?

(c) Describe S by an equation of the form $z = f(x, y)$.

(d) Using your answer in part (c), discuss whether S has a tangent plane at every point.

8. Verify that the image of the parametrized surface

$$\mathbf{X}(s, t) = (2 \sin s \cos t, 3 \sin s \sin t, \cos s),$$
$$0 \le s \le \pi, \quad 0 \le t \le 2\pi,$$

is an ellipsoid.

9. Verify that, for the torus of Example 5, the s-coordinate curve, when $t = t_0$, is a circle of radius $a + b \cos t_0$.

10. The surface in $\mathbf{R}^3$ parametrized by

$$\mathbf{X}(r, \theta) = (r \cos\theta, r \sin\theta, \theta), \quad r \ge 0, \quad -\infty < \theta < \infty,$$

is called a **helicoid**.

(a) Describe the r-coordinate curve when $\theta = \pi/3$. Give a general description of the r-coordinate curves.

(b) Describe the θ-coordinate curve when $r = 1$. Give a general description of the θ-coordinate curves.

(c) Sketch the graph of the helicoid (perhaps using a computer) for $0 \le r \le 1$, $0 \le \theta \le 4\pi$. Can you see why the surface is called a helicoid?

11. Given the sphere of radius 2 centered at $(2, -1, 0)$, find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

(a) by considering the sphere as the graph of the function

$$f(x, y) = \sqrt{4 - (x-2)^2 - (y+1)^2};$$

(b) by considering the sphere as a level surface of the function

$$F(x, y, z) = (x-2)^2 + (y+1)^2 + z^2;$$

(c) by considering the sphere as the surface parametrized by

$$\mathbf{X}(s, t) = (2 \sin s \cos t + 2, 2 \sin s \sin t - 1, 2 \cos s).$$

In Exercises 12–15, represent the given surface as a piecewise smooth parametrized surface.

12. The lower hemisphere $x^2 + y^2 + z^2 = 9$, including the equatorial circle.

13. The part of the cylinder $x^2 + z^2 = 4$ lying between $y = -1$ and $y = 3$.

14. The closed triangular region in $\mathbf{R}^3$ with vertices $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 5)$.

15. The hyperboloid $z^2 - x^2 - y^2 = 1$. (Hint: Use two maps to parametrize the surface.)

16. This problem concerns the parametrized surface $\mathbf{X}(s, t) = (s^3, t^3, st)$.

(a) Find an equation of the plane tangent to this surface at the point $(1, -1, -1)$.

(b) Is this surface smooth? Why or why not?

(c) Use a computer to graph this surface for $-1 \le s \le 1$, $-1 \le t \le 1$.

(d) Verify that this surface may also be described by the xyz-coordinate equation $z = \sqrt[3]{xy}$. Try using a computer to graph the surface when described in this form. Many software systems will have trouble, or will provide an incomplete graph, which is one reason why parametric descriptions of surfaces are desirable.

17. The surface given parametrically by $\mathbf{X}(s, t) = (st, t, s^2)$ is known as the **Whitney umbrella**.

(a) Verify that this surface may also be described by the xyz-coordinate equation $y^2 z = x^2$.

(b) Is $\mathbf{X}$ smooth?

(c) Use a computer to graph this surface for $-2 \le s \le 2$, $-2 \le t \le 2$.

(d) Some points (x, y, z) of the surface do not correspond to a single parameter point (s, t). Which ones? Explain how this relates to the graph.

(e) Give an equation of the plane tangent to this surface at the point $(2, 1, 4)$.

(f) Show that at the point $(0, 0, 1)$ on the image of $\mathbf{X}$ it's reasonable to conclude that there are *two* tangent planes. Give equations for them.

18. Let S be the surface defined as the graph of a function $f(x, y)$ of class C^1. Then Example 4 shows that S is also a parametrized surface. Show that formula (5) for the tangent plane to S at $(a, b, f(a, b))$ agrees with that of formula (4).

19. (a) Write a formula for the tangent plane to a surface described by the equation $y = g(x, z)$.

(b) Repeat part (a) for a surface described by the equation $x = h(y, z)$.

20. Suppose $\mathbf{X}: D \to \mathbf{R}^3$ is a parametrized surface that is smooth at $\mathbf{X}(s_0, t_0)$. Show how the definition of the derivative $D\mathbf{X}(s_0, t_0)$ can be used to give vector parametric equations for the plane tangent to $S = \mathbf{X}(D)$ at the point $\mathbf{X}(s_0, t_0)$.

21. Use the result of Exercise 20 to provide parametric equations for the plane tangent to the surface $\mathbf{X}(s, t) = (s, s^2 + t, t^2)$ at the point $(1, 0, 1)$. Verify that your answer is consistent with that of Exercise 5(c).

22. Use the parametrization in Example 3 to verify that the surface area of a cylinder of radius a and height h is $2\pi a h$.

23. Let D denote the unit disk in the st-plane. Let $\mathbf{X}: D \to \mathbf{R}^3$ be defined by $(s + t, s - t, s)$. Find the surface area of $\mathbf{X}(D)$.

24. Find the surface area of the helicoid

$$\mathbf{X}: D \to \mathbf{R}^3, \quad \mathbf{X}(r, \theta) = (r \cos\theta, r \sin\theta, \theta)$$

for $0 \le r \le 1$, $0 \le \theta \le 2\pi n$, where n is a positive integer.

25. A cylindrical hole of radius b is bored through a ball of radius a ($> b$) to form a ring. Find the outer surface area of the ring.

26. Find the area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies over the xy-plane.

27. Find the area of the surface cut from the paraboloid $z = 2x^2 + 2y^2$ by the planes $z = 2$ and $z = 8$.

28. Calculate the surface area of the portion of the plane $x + y + z = a$ cut out by the cylinder $x^2 + y^2 = a^2$ in two ways:

(a) by using formula (6);

(b) by using formula (9).

29. Let S be the surface defined by the equation $z = f(x, y)$. If $f_x^2 + f_y^2 = a$, where a is a positive constant, determine the surface area of the portion of S that lies over a region D in the xy-plane in terms of the area of D.

30. Let S be the surface defined by

$$z = \frac{1}{\sqrt{x^2 + y^2}} \quad \text{for } z \ge 1.$$

(a) Sketch the graph of this surface.

(b) Show that the volume of the region bounded by S and the plane $z = 1$ is finite. (You will need to use an improper integral.)

(c) Show that the surface area of S is infinite.

31. Find the surface area of the intersection of the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

32. Suppose that a surface is given in cylindrical coordinates by the equation $z = f(r, \theta)$, where (r, θ) varies through a region D in the $r\theta$-plane where r is nonnegative. Show that the surface area of the surface is given by

$$\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}\; r\, dr\, d\theta.$$

33. Suppose that a surface is given in spherical coordinates by the equation $\rho = f(\varphi, \theta)$, where (φ, θ) varies through a region D in the $\varphi\theta$-plane and $f(\varphi, \theta)$ is nonnegative. Show that the surface area of the surface is given by

$$\iint_D f(\varphi, \theta)$$
$$\times \sqrt{(f(\varphi, \theta)^2 + f_\varphi(\varphi, \theta)^2) \sin^2\varphi + f_\theta(\varphi, \theta)^2}\, d\varphi\, d\theta.$$

2 Surface Integrals

In this section, we will learn how to integrate both scalar-valued functions and vector fields along surfaces in $\mathbf{R}^3$. We begin by defining suitable integrals over parametrized surfaces and then establish that the particular choice of parametrization doesn't much matter—that, really, only the underlying surface is important, and possibly the orientation.

From the change of variables theorem, it follows that

$$\iint_{\mathbf{Y}} f\, dS = \iint_{D_1} f(\mathbf{X}(u, v))\, \|\mathbf{N_X}(u, v)\|\, du\, dv = \iint_{\mathbf{X}} f\, dS,$$

by Definition 2.1. ■

Proof of Theorem 2.5 This result can be established along the lines of the previous proof. Beginning with Definition 2.2 and using the lemma just established, we have

$$\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D_2} \mathbf{F}(\mathbf{Y}(s, t)) \cdot \mathbf{N_Y}(s, t)\, ds\, dt$$

$$= \iint_{D_2} \mathbf{F}(\mathbf{X}(\mathbf{H}(s, t))) \cdot \frac{\partial(u, v)}{\partial(s, t)} \mathbf{N_X}(u(s, t), v(s, t))\, ds\, dt.$$

Therefore,

$$\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = \pm \iint_{D_2} \mathbf{F}(\mathbf{X}(\mathbf{H}(s, t))) \cdot \mathbf{N_X}(u(s, t), v(s, t)) \left| \frac{\partial(u, v)}{\partial(s, t)} \right| ds\, dt,$$

where we take the "+" sign if $\mathbf{Y}$ is an orientation-preserving reparametrization of $\mathbf{X}$ (since the Jacobian $\partial(u, v)/\partial(s, t)$ is positive and hence equal to its absolute value) and the "−" sign if $\mathbf{Y}$ is orientation-reversing. By the change of variables theorem, this last expression is equal to

$$\pm \iint_{D_1} \mathbf{F}(\mathbf{X}(u, v)) \cdot \mathbf{N_X}(u, v)\, du\, dv = \pm \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S},$$

by Definition 2.2. ■

2 Exercises

1. Let $\mathbf{X}(s, t) = (s, s + t, t)$, $0 \le s \le 1$, $0 \le t \le 2$. Find

$$\iint_{\mathbf{X}} (x^2 + y^2 + z^2)\, dS.$$

2. Let $D = \{(s, t) \mid s^2 + t^2 \le 1,\ s \ge 0,\ t \ge 0\}$ and let $\mathbf{X}\colon D \to \mathbf{R}^3$ be defined by $\mathbf{X}(s, t) = (s + t, s - t, st)$.

(a) Determine $\iint_{\mathbf{X}} f\, dS$, where $f(x, y, z) = 4$.

(b) Find the value of $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

3. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the surface S consisting of the triangular region of the plane $2x - 2y + z = 2$ that is cut out by the coordinate planes. Use an upward-pointing normal to orient S.

4. This problem concerns the two surfaces given parametrically as

$$\mathbf{X}(s, t) = (s \cos t, s \sin t, 3s^2),$$
$$0 \le s \le 2, 0 \le t \le 2\pi.$$

and

$$\mathbf{Y}(s, t) = (2s \cos t, 2s \sin t, 12s^2),$$
$$0 \le s \le 1, 0 \le t \le 4\pi.$$

(a) Show that the images of $\mathbf{X}$ and $\mathbf{Y}$ are the same. (Hint: Give equations in x, y, and z for the surfaces in $\mathbf{R}^3$ parametrized by $\mathbf{X}$ and $\mathbf{Y}$.)

(b) Calculate $\iint_{\mathbf{X}} (y\,\mathbf{i} - x\,\mathbf{j} + z^2\,\mathbf{k}) \cdot d\mathbf{S}$ and $\iint_{\mathbf{Y}} (y\,\mathbf{i} - x\,\mathbf{j} + z^2\,\mathbf{k}) \cdot d\mathbf{S}$. Reconcile your answers.

5. Find $\iint_S x^2\, dS$, where S is the surface of the cube $[-2, 2] \times [-2, 2] \times [-2, 2]$.

6. Find $\iint_S (x^2 + y^2)\, dS$, where S is the lateral surface of the cylinder of radius a and height h whose axis is the z-axis.

7. Let S be a sphere of radius a.

(a) Find $\iint_S (x^2 + y^2 + z^2)\, dS$.

(b) Use symmetry and part (a) to easily find $\iint_S y^2\, dS$.

From the change of variables theorem, it follows that

$$\iint_{\mathbf{Y}} f\,dS = \iint_{D_1} f(\mathbf{X}(u,v))\,\|\mathbf{N}_{\mathbf{X}}(u,v)\|\,du\,dv = \iint_{\mathbf{X}} f\,dS,$$

by Definition 2.1. ■

Proof of Theorem 2.5 This result can be established along the lines of the previous proof. Beginning with Definition 2.2 and using the lemma just established, we have

$$\iint_{\mathbf{Y}} \mathbf{F}\cdot d\mathbf{S} = \iint_{D_2} \mathbf{F}(\mathbf{Y}(s,t))\cdot \mathbf{N}_{\mathbf{Y}}(s,t)\,ds\,dt$$

$$= \iint_{D_2} \mathbf{F}(\mathbf{X}(\mathbf{H}(s,t)))\cdot \frac{\partial(u,v)}{\partial(s,t)}\mathbf{N}_{\mathbf{X}}(u(s,t),v(s,t))\,ds\,dt.$$

Therefore,

$$\iint_{\mathbf{Y}} \mathbf{F}\cdot d\mathbf{S} = \pm\iint_{D_2} \mathbf{F}(\mathbf{X}(\mathbf{H}(s,t)))\cdot \mathbf{N}_{\mathbf{X}}(u(s,t),v(s,t))\left|\frac{\partial(u,v)}{\partial(s,t)}\right| ds\,dt,$$

where we take the "+" sign if $\mathbf{Y}$ is an orientation-preserving reparametrization of $\mathbf{X}$ (since the Jacobian $\partial(u,v)/\partial(s,t)$ is positive and hence equal to its absolute value) and the "−" sign if $\mathbf{Y}$ is orientation-reversing. By the change of variables theorem, this last expression is equal to

$$\pm\iint_{D_1} \mathbf{F}(\mathbf{X}(u,v))\cdot \mathbf{N}_{\mathbf{X}}(u,v)\,du\,dv = \pm\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S},$$

by Definition 2.2. ■

2 Exercises

1. Let $\mathbf{X}(s,t) = (s, s+t, t),\ 0 \le s \le 1,\ 0 \le t \le 2.$ Find

$$\iint_{\mathbf{X}} (x^2+y^2+z^2)\,dS.$$

2. Let $D = \{(s,t) \mid s^2+t^2 \le 1,\ s \ge 0,\ t \ge 0\}$ and let $\mathbf{X}\colon D \to \mathbf{R}^3$ be defined by $\mathbf{X}(s,t) = (s+t, s-t, st)$.
(a) Determine $\iint_{\mathbf{X}} f\,dS$, where $f(x,y,z) = 4$.
(b) Find the value of $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

3. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the surface S consisting of the triangular region of the plane $2x - 2y + z = 2$ that is cut out by the coordinate planes. Use an upward-pointing normal to orient S.

4. This problem concerns the two surfaces given parametrically as

$$\mathbf{X}(s,t) = (s\cos t, s\sin t, 3s^2),$$
$$0 \le s \le 2, 0 \le t \le 2\pi.$$

and

$$\mathbf{Y}(s,t) = (2s\cos t, 2s\sin t, 12s^2),$$
$$0 \le s \le 1, 0 \le t \le 4\pi.$$

(a) Show that the images of $\mathbf{X}$ and $\mathbf{Y}$ are the same. (Hint: Give equations in x, y, and z for the surfaces in $\mathbf{R}^3$ parametrized by $\mathbf{X}$ and $\mathbf{Y}$.)
(b) Calculate $\iint_{\mathbf{X}}(y\,\mathbf{i} - x\,\mathbf{j} + z^2\,\mathbf{k})\cdot d\mathbf{S}$ and $\iint_{\mathbf{Y}}(y\,\mathbf{i} - x\,\mathbf{j} + z^2\,\mathbf{k})\cdot d\mathbf{S}$. Reconcile your answers.

5. Find $\iint_S x^2\,dS$, where S is the surface of the cube $[-2,2]\times[-2,2]\times[-2,2]$.

6. Find $\iint_S (x^2+y^2)\,dS$, where S is the lateral surface of the cylinder of radius a and height h whose axis is the z-axis.

7. Let S be a sphere of radius a.
(a) Find $\iint_S (x^2+y^2+z^2)\,dS$.
(b) Use symmetry and part (a) to easily find $\iint_S y^2\,dS$.

8. Let S denote the sphere $x^2+y^2+z^2 = a^2$.
(a) Use symmetry considerations to evaluate $\iint_S x\,dS$ *without* resorting to parametrizing the sphere.
(b) Let $\mathbf{F} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. Use symmetry to determine $\iint_S \mathbf{F}\cdot d\mathbf{S}$ without parametrizing the sphere.

9. Let S denote the surface of the cylinder $x^2+y^2 = 4$, $-2 \le z \le 2$, and consider the surface integral

$$\iint_S (z - x^2 - y^2)\,dS.$$

(a) Use an appropriate parametrization of S to calculate the value of the integral.
(b) Now use geometry and symmetry to evaluate the integral without resorting to a parametrization of the surface.

In Exercises 10–18, let S denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by the equation $x^2+y^2 = 9$. Orient S with outward normals. Determine the indicated scalar and vector surface integrals.

10. $\iint_S z\,dS$

11. $\iint_S y\,dS$

12. $\iint_S xyz\,dS$

13. $\iint_S x^2 dS$

14. $\iint_S (x\,\mathbf{i}+y\,\mathbf{j})\cdot d\mathbf{S}$

15. $\iint_S z\,\mathbf{k}\cdot d\mathbf{S}$

16. $\iint_S y^3\,\mathbf{i}\cdot d\mathbf{S}$

17. $\iint_S (-y\,\mathbf{i}+x\,\mathbf{j})\cdot d\mathbf{S}$

18. $\iint_S x^2\,\mathbf{i}\cdot d\mathbf{S}$

In Exercises 19–22, find the flux of the given vector field $\mathbf{F}$ across the upper hemisphere $x^2+y^2+z^2 = a^2$, $z \ge 0$. Orient the hemisphere with an upward-pointing normal.

19. $\mathbf{F} = y\,\mathbf{j}$

20. $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$

21. $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} - \mathbf{k}$

22. $\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$

23. Let S be the parametrized helicoid $\mathbf{X}(s,t) = (s\cos t, s\sin t, t)$, with $0 \le s \le 2, 0 \le t \le 2\pi$. Determine the flux of $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + z^3\,\mathbf{k}$ across S.

24. Let $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + z^2\,\mathbf{k}$. Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the portion of the cone $x^2+y^2 = z^2$ between the planes $z = -2$, and $z = 1$, oriented with outward-pointing normal.

25. Find the flux of $\mathbf{F} = y^3z\,\mathbf{i} - xy\,\mathbf{j} + (x+y+z)\,\mathbf{k}$ across the portion of the surface $z = ye^x$ lying over the unit square $[0,1]\times[0,1]$ in the xy-plane, oriented by upward normal.

26. Let S denote the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$ oriented by outward normal, and let $\mathbf{F} = x^2\,\mathbf{i} + 4z\,\mathbf{j} + (y-x)\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across S.

27. Let S be the funnel-shaped surface defined by $x^2 + y^2 = z^2$ for $1 \le z \le 9$ and $x^2+y^2 = 1$ for $0 \le z \le 1$.
(a) Sketch S.
(b) Determine outward-pointing unit normal vectors to S.
(c) Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is oriented by outward normals.

28. The glass dome of a futuristic greenhouse is shaped like the surface $z = 8 - 2x^2 - 2y^2$. The greenhouse has a flat dirt floor at $z = 0$. Suppose that the temperature T, at points in and around the greenhouse, varies as

$$T(x,y,z) = x^2 + y^2 + 3(z-2)^2.$$

Then the temperature gives rise to a **heat flux density field H** given by $\mathbf{H} = -k\nabla T$. (Here k is a positive constant that depends on the insulating properties of the particular medium.) Find the total heat flux outward across the dome and the surface of the ground if $k = 1$ on the glass and $k = 3$ on the ground.

29. The surface given by $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$, where

$$\begin{cases} x = \left(a + \cos\dfrac{s}{2}\,\sin t - \sin\dfrac{s}{2}\,\sin 2t\right)\cos s \\ y = \left(a + \cos\dfrac{s}{2}\,\sin t - \sin\dfrac{s}{2}\,\sin 2t\right)\sin s\ , \\ z = \sin\dfrac{s}{2}\,\sin t + \cos\dfrac{s}{2}\,\sin 2t \end{cases}$$

a is a positive constant, and $0 \le s \le 2\pi, 0 \le t \le 2\pi$, is known as a **Klein bottle**.
(a) Use a computer to plot this surface for $a = 2$.
(b) Determine (and describe) the s-coordinate curve at $t = 0$.
(c) Calculate the standard normal vector $\mathbf{N}$ along the s-coordinate curve at $t = 0$ (i.e., find $\mathbf{N}(s,0)$). Note that $\mathbf{X}(0,0) = \mathbf{X}(2\pi,0)$. By comparing $\mathbf{N}(0,0)$ and $\mathbf{N}(2\pi,0)$, comment regarding the orientability of the Klein bottle. (See Example 8.)