The standard way to represent a polynomial in P_n is in terms of the functions $1, x, x^2, \ldots, x^{n-1}$, and consequently, the standard basis for P_n is $\{1, x, x^2, \ldots, x^{n-1}\}$.

Although these standard bases appear to be the simplest and most natural to use, they are not the most appropriate bases for many applied problems. Indeed, the key to solving many applied problems is to switch from one of the standard bases to a basis that is in some sense natural for the particular application. Once the application is solved in terms of the new basis, it is a simple matter to switch back and represent the solution in terms of the standard basis. In the next section, we will learn how to switch from one basis to another.

SECTION 4 EXERCISES

1. In Exercise 1 of Section 3, indicate whether the given vectors form a basis for $\mathbb{R}^2$.

2. In Exercise 2 of Section 3, indicate whether the given vectors form a basis for $\mathbb{R}^3$.

3. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

(a) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for $\mathbb{R}^2$.

(b) Why must $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ be linearly dependent?

(c) What is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

4. Given the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$$

what is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

5. Let

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly dependent.

(b) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

(c) What is the dimension of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$?

(d) Give a geometric description of $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

6. In Exercise 2 of Section 2, some of the sets formed subspaces of $\mathbb{R}^3$. In each of these cases, find a basis for the subspace and determine its dimension.

7. Find a basis for the subspace S of $\mathbb{R}^4$ consisting of all vectors of the form $(a + b, a - b + 2c, b, c)^T$, where a, b, and c are all real numbers. What is the dimension of S?

8. Given $\mathbf{x}_1 = (1,\ 1,\ 1)^T$ and $\mathbf{x}_2 = (3,\ -1,\ 4)^T$:

(a) Do $\mathbf{x}_1$ and $\mathbf{x}_2$ span $\mathbb{R}^3$? Explain.

(b) Let $\mathbf{x}_3$ be a third vector in $\mathbb{R}^3$ and set $X = (\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3)$. What condition(s) would X have to satisfy in order for $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ to form a basis for $\mathbb{R}^3$?

(c) Find a third vector $\mathbf{x}_3$ that will extend the set $\{\mathbf{x}_1, \mathbf{x}_2\}$ to a basis for $\mathbb{R}^3$.

9. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be linearly independent vectors in $\mathbb{R}^3$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^2$.

(a) Describe geometrically $\text{Span}(\mathbf{a}_1, \mathbf{a}_2)$.

(b) If $A = (\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{b} = A\mathbf{x}$, then what is the dimension of $\text{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b})$? Explain.

10. The vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix},$$

$$\mathbf{x}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \quad \mathbf{x}_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

span $\mathbb{R}^3$. Pare down the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ to form a basis for $\mathbb{R}^3$.

11. Let S be the subspace of P_3 consisting of all polynomials of the form $ax^2 + bx + 2a + 3b$. Find a basis for S.

12. In Exercise 3 of Section 2, some of the sets formed subspaces of $\mathbb{R}^{2\times 2}$. In each of these cases, find a basis for the subspace and determine its dimension.

13. In $C[-\pi, \pi]$, find the dimension of the subspace spanned by 1, $\cos 2x$, and $\cos^2 x$.

14. In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:

(a) $x, x - 1, x^2 + 1$

(b) $x, x-1, x^2+1, x^2-1$

(c) $x^2, x^2-x-1, x+1$ **(d)** $2x, x-2$

15. Let S be the subspace of P_3 consisting of all polynomials $p(x)$ such that $p(0)=0$, and let T be the subspace of all polynomials $q(x)$ such that $q(1)=0$. Find bases for

(a) S **(b)** T **(c)** $S \cap T$

16. In $\mathbb{R}^4$, let U be the subspace of all vectors of the form $(u_1, u_2, 0, 0)^T$, and let V be the subspace of all vectors of the form $(0, v_2, v_3, 0)^T$. What are the dimensions of U, V, $U \cap V$, $U+V$? Find a basis for each of these four subspaces. (See Exercises 20 and 22 of Section 2.)

17. Is it possible to find a pair of two-dimensional subspaces U and V of $\mathbb{R}^3$ such that $U \cap V = \{\mathbf{0}\}$? Prove your answer. Give a geometrical interpretation of your conclusion. [*Hint*: Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be bases for U and V, respectively. Show that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are linearly dependent.]

18. Show that if U and V are subspaces of $\mathbb{R}^n$ and $U \cap V = \{\mathbf{0}\}$, then

$$\dim(U+V) = \dim U + \dim V$$

5 Change of Basis

Many applied problems can be simplified by changing from one coordinate system to another. Changing coordinate systems in a vector space is essentially the same as changing from one basis to another. For example, in describing the motion of a particle in the plane at a particular time, it is often convenient to use a basis for $\mathbb{R}^2$ consisting of a unit tangent vector $\mathbf{T}$ and a unit normal vector $\mathbf{N}$ instead of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

In this section, we discuss the problem of switching from one coordinate system to another. We will show that this can be accomplished by multiplying a given coordinate vector $\mathbf{x}$ by a nonsingular matrix S. The product $\mathbf{y} = S\mathbf{x}$ will be the coordinate vector for the new coordinate system.

Changing Coordinates in $\mathbb{R}^2$

The standard basis for $\mathbb{R}^2$ is $\{\mathbf{e}_1, \mathbf{e}_2\}$. Any vector $\mathbf{x}$ in $\mathbb{R}^2$ can be expressed as a linear combination

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

The scalars x_1 and x_2 can be thought of as the *coordinates* of $\mathbf{x}$ with respect to the standard basis. Actually, for any basis $\{\mathbf{y}, \mathbf{z}\}$ for $\mathbb{R}^2$, it follows from Theorem 3.2 that a given vector $\mathbf{x}$ can be represented uniquely as a linear combination

$$\mathbf{x} = \alpha\mathbf{y} + \beta\mathbf{z}$$

The scalars α and β are the coordinates of $\mathbf{x}$ with respect to the basis $\{\mathbf{y}, \mathbf{z}\}$. Let us order the basis elements so that $\mathbf{y}$ is considered the first basis vector and $\mathbf{z}$ is considered the second, and denote the ordered basis by $[\mathbf{y}, \mathbf{z}]$. We can then refer to the vector $(\alpha, \beta)^T$ as the *coordinate vector* of $\mathbf{x}$ with respect to $[\mathbf{y}, \mathbf{z}]$. Note that, if we reverse the order of the basis vectors and take $[\mathbf{z}, \mathbf{y}]$, then we must also reorder the coordinate vector. The coordinate vector of $\mathbf{x}$ with respect to $[\mathbf{z}, \mathbf{y}]$ will be $(\beta, \alpha)^T$. When we refer to a basis using subscripts, such as $\{\mathbf{u}_1, \mathbf{u}_2\}$, the subscripts assign an ordering to the basis vectors.

EXAMPLE 1 Let $\mathbf{y} = (2, 1)^T$ and $\mathbf{z} = (1, 4)^T$. The vectors $\mathbf{y}$ and $\mathbf{z}$ are linearly independent and hence they form a basis for $\mathbb{R}^2$. The vector $\mathbf{x} = (7, 7)^T$ can be written as a linear

Since S is nonsingular, $\mathbf{x}$ must equal $\mathbf{0}$. Therefore, $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent and hence they form a basis for V. The matrix S is the transition matrix corresponding to the change from the ordered basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ to $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

In many applied problems, it is important to use the right type of basis for the particular application. You may consider a number of applications involving the *eigenvalues* and *eigenvectors* associated with an $n \times n$ matrix A. The key to solving these types of problems is to switch to a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

SECTION 5 EXERCISES

1. For each of the following, find the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$:

(a) $\mathbf{u}_1 = (1, 1)^T, \quad \mathbf{u}_2 = (-1, 1)^T$

(b) $\mathbf{u}_1 = (1, 2)^T, \quad \mathbf{u}_2 = (2, 5)^T$

(c) $\mathbf{u}_1 = (0, 1)^T, \quad \mathbf{u}_2 = (1, 0)^T$

2. For each of the ordered bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ in Exercise 1, find the transition matrix corresponding to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

3. Let $\mathbf{v}_1 = (3, 2)^T$ and $\mathbf{v}_2 = (4, 3)^T$. For each ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given in Exercise 1, find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

4. Let $E = [(5, 3)^T, (3, 2)^T]$ and let $\mathbf{x} = (1, 1)^T$, $\mathbf{y} = (1, -1)^T$, and $\mathbf{z} = (10, 7)^T$. Determine the values of $[\mathbf{x}]_E$, $[\mathbf{y}]_E$, and $[\mathbf{z}]_E$.

5. Let $\mathbf{u}_1 = (1, 1, 1)^T$, $\mathbf{u}_2 = (1, 2, 2)^T$, $\mathbf{u}_3 = (2, 3, 4)^T$.

(a) Find the transition matrix corresponding to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(b) Find the coordinates of each of the following vectors with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$:

(i) $(3, 2, 5)^T$ **(ii)** $(1, 1, 2)^T$

(iii) $(2, 3, 2)^T$

6. Let $\mathbf{v}_1 = (4, 6, 7)^T$, $\mathbf{v}_2 = (0, 1, 1)^T$, $\mathbf{v}_3 = (0, 1, 2)^T$, and let $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ be the vectors given in Exercise 5.

(a) Find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(b) If $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$, determine the coordinates of $\mathbf{x}$ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

7. Given

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad S = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix}$$

find vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ so that S will be the transition matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

8. Given

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad S = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$$

find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that S will be the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

9. Let $[x, 1]$ and $[2x - 1, 2x + 1]$ be ordered bases for P_2.

(a) Find the transition matrix representing the change in coordinates from $[2x - 1, 2x + 1]$ to $[x, 1]$.

(b) Find the transition matrix representing the change in coordinates from $[x, 1]$ to $[2x - 1, 2x + 1]$.

10. Find the transition matrix representing the change of coordinates on P_3 from the ordered basis $[1, x, x^2]$ to the ordered basis

$$[1, 1 + x, 1 + x + x^2]$$

11. Let $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $F = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be two ordered bases for $\mathbb{R}^n$, and set

$$U = (\mathbf{u}_1, \ldots, \mathbf{u}_n), \quad V = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

Show that the transition matrix from E to F can be determined by calculating the reduced row echelon form of $(V \mid U)$.

EXAMPLE 5 Find the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

Solution

The subspace $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ is the same as the column space of the matrix

$$X = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

The row echelon form of X is

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns $\mathbf{x}_1$ and $\mathbf{x}_2$ of X will form a basis for the column space of X. Thus, $\dim \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = 2$. ■

SECTION 6 EXERCISES

1. For each of the following matrices, find a basis for the row space, a basis for the column space, and a basis for the null space:

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

2. In each of the following, determine the dimension of the subspace of $\mathbb{R}^3$ spanned by the given vectors:

(a) $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

3. Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix}$$

(a) Compute the reduced row echelon form U of A. Which column vectors of U correspond to the free variables? Write each of these vectors as a linear combination of the column vectors corresponding to the lead variables.

(b) Which column vectors of A correspond to the lead variables of U? These column vectors form a basis for the column space of A. Write each of the remaining column vectors of A as a linear combination of these basis vectors.

4. For each of the following choices of A and $\mathbf{b}$, determine whether $\mathbf{b}$ is in the column space of A and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent:

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(e) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$

(f) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}$

5. For each consistent system in Exercise 4, determine whether there will be one or infinitely many solutions by examining the column vectors of the coefficient matrix A.

6. How many solutions will the linear system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is in the column space of A and the column vectors of A are linearly dependent? Explain.

7. Let A be an $6 \times n$ matrix of rank r and let $\mathbf{b}$ be a vector in $\mathbb{R}^6$. For each pair of values of r and n that follow, indicate the possibilities as to the number of solutions one could have for the linear system $A\mathbf{x} = \mathbf{b}$. Explain your answers.

(a) $n = 7, r = 5$ **(b)** $n = 7, r = 6$

(c) $n = 5, r = 5$ **(d)** $n = 5, r = 4$

8. Let A be an $m \times n$ matrix with $m > n$. Let $\mathbf{b} \in \mathbb{R}^m$ and suppose that $N(A) = \{\mathbf{0}\}$.

(a) What can you conclude about the column vectors of A? Are they linearly independent? Do they span $\mathbb{R}^m$? Explain.

(b) How many solutions will the system $A\mathbf{x} = \mathbf{b}$ have if $\mathbf{b}$ is not in the column space of A? How many solutions will there be if $\mathbf{b}$ is in the column space of A? Explain.

9. Let A and B be 6×5 matrices. If $\dim N(A) = 2$, what is the rank of A? If the rank of B is 4, what is the dimension of $N(B)$?

10. Let A be an $m \times n$ matrix whose rank is equal to n. If $A\mathbf{c} = A\mathbf{d}$, does this imply that $\mathbf{c}$ must be equal to $\mathbf{d}$? What if the rank of A is less than n? Explain your answers.

11. Let A be an $m \times n$ matrix. Prove that

$$\text{rank}(A) \leq \min(m, n)$$

12. Let A and B be row-equivalent matrices.

(a) Show that the dimension of the column space of A equals the dimension of the column space of B.

(b) Are the column spaces of the two matrices necessarily the same? Justify your answer.

13. Let A be a 4×3 matrix and suppose that the vectors

$$\mathbf{z}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{z}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

form a basis for $N(A)$. If $\mathbf{b} = \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$, find all solutions of the system $A\mathbf{x} = \mathbf{b}$.

14. Let A be a 4×4 matrix with reduced row echelon form given by

$$U = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If

$$\mathbf{a}_1 = \begin{pmatrix} -3 \\ 5 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 4 \\ -3 \\ 7 \\ -1 \end{pmatrix}$$

find $\mathbf{a}_3$ and $\mathbf{a}_4$.

15. Let A be a 4×5 matrix and let U be the reduced row echelon form of A. If

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \\ -2 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) find a basis for $N(A)$.

(b) given that $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} 0 \\ 5 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(i) find all solutions to the system.

(ii) determine the remaining column vectors of A.

16. Let A be a 5×8 matrix with rank equal to 5 and let $\mathbf{b}$ be any vector in $\mathbb{R}^5$. Explain why the system $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions.

17. Let A be a 4×5 matrix. If $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$ are linearly independent and

$$\mathbf{a}_3 = \mathbf{a}_1 + 2\mathbf{a}_2, \quad \mathbf{a}_5 = 2\mathbf{a}_1 - \mathbf{a}_2 + 3\mathbf{a}_4$$

determine the reduced row echelon form of A.

18. Let A be a 5×3 matrix of rank 3 and let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a basis for $\mathbb{R}^3$.

(a) Show that $N(A) = \{\mathbf{0}\}$.

(b) Show that if $\mathbf{y}_1 = A\mathbf{x}_1$, $\mathbf{y}_2 = A\mathbf{x}_2$, $\mathbf{y}_3 = A\mathbf{x}_3$, then $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent.

(c) Do the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ from part (b) form a basis for $\mathbb{R}^5$? Explain.

19. Let A be an $m \times n$ matrix with rank equal to n. Show that if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = A\mathbf{x}$, then $\mathbf{y} \neq \mathbf{0}$.

20. Prove that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of $(A \mid \mathbf{b})$ equals the rank of A.

21. Let A and B be $m \times n$ matrices. Show that

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$$

22. Let A be an $m \times n$ matrix.

(a) Show that if B is a nonsingular $m \times m$ matrix, then BA and A have the same null space and hence the same rank.

(b) Show that if C is a nonsingular $n \times n$ matrix, then AC and A have the same rank.

23. Prove Corollary 6.4.

24. Show that if A and B are $n \times n$ matrices and $N(A - B) = \mathbb{R}^n$, then $A = B$.

25. Let A and B be $n \times n$ matrices.

(a) Show that $AB = O$ if and only if the column space of B is a subspace of the null space of A.

(b) Show that if $AB = O$, then the sum of the ranks of A and B cannot exceed n.

26. Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x}_0$ be a particular solution of the system $A\mathbf{x} = \mathbf{b}$. Prove the following:

(a) A vector $\mathbf{y}$ in $\mathbb{R}^n$ will be a solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, where $\mathbf{z} \in N(A)$.

(b) If $N(A) = \{\mathbf{0}\}$, then the solution $\mathbf{x}_0$ is unique.

27. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $A = \mathbf{x}\mathbf{y}^T$.

(a) Show that $\{\mathbf{x}\}$ is a basis for the column space of A and that $\{\mathbf{y}^T\}$ is a basis for the row space of A.

(b) What is the dimension of $N(A)$?

28. Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C = AB$. Show that

(a) the column space of C is a subspace of the column space of A.

(b) the row space of C is a subspace of the row space of B.

(c) $\text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

29. Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C = AB$. Show that

(a) if A and B both have linearly independent column vectors, then the column vectors of C will also be linearly independent.

(b) if A and B both have linearly independent row vectors, then the row vectors of C will also be linearly independent.

[*Hint*: Apply part (a) to C^T.]

30. Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C = AB$. Show that

(a) if the column vectors of B are linearly dependent, then the column vectors of C must be linearly dependent.

(b) if the row vectors of A are linearly dependent, then the row vectors of C are linearly dependent.

[*Hint*: Apply part (a) to C^T.]

31. An $m \times n$ matrix A is said to have a *right inverse* if there exists an $n \times m$ matrix C such that $AC = I_m$. A is said to have a *left inverse* if there exists an $n \times m$ matrix D such that $DA = I_n$.

(a) Show that if A has a right inverse, then the column vectors of A span $\mathbb{R}^m$.

(b) Is it possible for an $m \times n$ matrix to have a right inverse if $n < m$? $n \geq m$? Explain.

32. Prove: If A is an $m \times n$ matrix and the column vectors of A span $\mathbb{R}^m$, then A has a right inverse. [*Hint*: Let $\mathbf{e}_j$ denote the jth column of I_m, and solve $A\mathbf{x} = \mathbf{e}_j$ for $j = 1, \ldots, m$.]

33. Show that a matrix B has a left inverse if and only if B^T has a right inverse.

34. Let B be an $n \times m$ matrix whose columns are linearly independent. Show that B has a left inverse.

35. Prove that if a matrix B has a left inverse, then the columns of B are linearly independent.

36. Show that if a matrix U is in row echelon form, then the nonzero row vectors of U form a basis for the row space of U.