Math. 401, Sec. 501 Spring, 2016

Homework 3, due Feb 11

1. Verify that the following are Sturm–Liouville eigenvalue problems, and specify whether they are regular or singular:

   (i) $f''(x) + \lambda f(x) = 0, \quad 0 < x < 1,$
   $f(0) + 2f'(0) = 0, \quad f'(1) = 0;$

   (iii) $2f''(x) + f'(x) + \lambda f(x) = 0, \quad 0 < x < \infty,$
   $f(0) = 0, \quad f(x), f'(x)$ are bounded as $x \to \infty$;

2. Compute the eigenvalues and eigenfunctions of the following S–L problems:

   (i) $f''(x) + \lambda f(x) = 0, \quad 0 < x < \pi,$
   $f(0) = 0, \quad f'(\pi) = 0;$

   (iii) $f''(x) + \lambda f(x) = 0, \quad 0 < x < 1,$
   $f'(0) - f(0) = 0, \quad f(1) = 0;$

   (v) $f''(x) + \lambda f(x) = 0, \quad 0 < x < 1,$
   $f'(0) - f(0) = 0, \quad f'(1) = 0;$

3. Find the Generalized Fourier series expansions of the following functions in the eigenfunctions of the problem solved in class (i.e. the example (3.10) (see also (3.16) in the book).

$$\text{(i)} \quad u(x) \equiv 1, \quad 0 \le x \le \pi;$$

$$\text{(iii)} \quad u(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1 - x, & \frac{1}{2} < x \le 1. \end{cases}$$

Answers

1. (i) Regular. (iii) Singular.

2. (i) $\lambda_n = (2n-1)^2/4, \ f_n(x) = \sin((2n-1)x/2), \ n = 1, 2, \ldots$

   (iii) $\lambda_n = \zeta_n^2$, where ζ_n are the roots of the equation $\tan \zeta = -\zeta$,
   $f_n(x) = \zeta_n \cos(\zeta_n x) + \sin(\zeta_n x), \ n = 1, 2, \ldots$

   (v) $\lambda_n = \zeta_n^2$, where ζ_n are the roots of the equation $\cot \zeta = \zeta$,
   $f_n(x) = \zeta_n \cos(\zeta_n x) + \sin(\zeta_n x), \ n = 1, 2, \ldots$

3. (i) $u(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - (-1)^n] \sin(nx).$

   (iii) $u(x) \sim \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2} \sin(n\pi x).$