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Analytical and numerical studies of a singularly perturbed Boussinesq equation

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Abstract

We study the singularly perturbed (sixth-order) Boussinesq equation recently introduced by Daripa and Hua [Appl. Math. Comput. 101 (1999) 159]. Motivated by their work, we formally derive this equation from two-dimensional potential flow equations governing the small amplitude long capillary-gravity waves on the surface of shallow water for Bond number very close to but less than $1/3$. On the basis of far-field analyses and heuristic arguments, we show that the traveling wave solutions of this equation are weakly non-local solitary waves characterized by small amplitude fast oscillations in the far-field. We review various analytical and numerical methods originally devised to obtain this type of weakly non-local solitary wave solutions of the singularly perturbed (fifth-order) KdV equation. Using these methods, we obtain weakly non-local solitary wave solutions of the singularly perturbed (sixth-order) Boussinesq equation and provide estimates of the amplitude of oscillations which persist in the far-field. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

The study of wave propagation on the surface of water and, in particular, solitary waves has been a subject of considerable theoretical and practical importance for over a century and a half. In 1872, Boussinesq [12] derived a

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model equation for propagation of water waves from Euler's equations of motion for two-dimensional potential flow beneath a free surface by introducing appropriate approximations for small amplitude long waves. This equation is now known as the classical Boussinesq equation. Later, Korteweg–deVries [25] derived a simpler model equation, now known as the classical KdV equation, by using far-field analysis in addition to Boussinesq's approximations (see [24] for more detail). Both the Boussinesq and KdV equations possess solitary wave solutions. The Boussinesq equation describes bi-directional wave propagation, whereas, the KdV equation describes uni-directional wave propagation.

The problem of existence of solitary wave solutions to the full non-linear water wave equations with or without surface tension has been the subject of considerable investigations in the last two decades. The solutions to the full water wave equations are characterized by the non-dimensional surface tension parameter τ , called the Bond number. This parameter is defined as follows: $\tau = \Gamma / \rho g h_0^2$, where Γ is the surface tension coefficient, ρ is the density of the water, g is the acceleration due to gravity and h_0 is the height of the undisturbed water surface.

The existence of solitary wave solutions to the full non-linear water wave equations without surface tension effects has been proved by Amick and Toland [6] and Beale [8]. Hunter and Vanden-Broeck [23] solved the full non-linear water wave equations with surface tension effects included numerically. Their computed solitary wave solutions for $\tau > 1/3$ agreed with the KdV solitary waves. But, they were unable to compute solitary wave solutions for $0 < \tau < 1/3$. On the basis of their computations, they concluded that the classical KdV equation is perhaps not a good model for the full water wave equations for $0 < \tau < 1/3$. Later, through the analyses of full non-linear water wave equations, Amick and Kirchgässner [3,4] and Sachs [29] proved the existence of solitary wave solutions to the water wave problem for $\tau > 1/3$, and Beale [9], Sun [32], Sun and Shen [33] and Vanden-Broeck [34] proved the non-existence of solitary wave solutions and existence of generalized solitary wave solutions with oscillatory tails to the water wave problem for $0 < \tau < 1/3$.

Hunter and Scheurle [22] derived a new model equation, namely the singularly perturbed (fifth-order) KdV equation, to describe the uni-directional propagation of small amplitude long capillary-gravity waves on the surface of shallow water for $\tau \uparrow 1/3$ (i.e. Bond number τ is less than but very close to $1/3$). The solutions they found were arbitrary small perturbations of KdV solitary waves. These were not classical solitary waves because of the presence of small amplitude fast oscillations at distances far from the core of the waves and extending up to infinity. These waves are known as weakly non-local solitary waves.

The fifth-order KdV equation has gained much popularity over last decade. With various objectives in mind, this equation has been studied analytically by

Akylas and Yang [2], Amick and McLeod [5], Amick and Toland [7], Benilov et al. [11], Boyd [14], Byatt-Smith [16], Eckhaus [19,20], Grimshaw and Joshi [21], Kichenassamy and Olver [26] and Pomeau et al. [28], and numerically by Boyd [15]. Their work suggests the non-existence of classical local solitary wave solutions and the existence of weakly non-local solitary wave solutions for this equation. The corresponding non-local internal solitary waves for mode number greater than 1 were studied analytically by Akylas and Grimshaw [1] and numerically by Vanden-Broeck and Turner [35].

A perturbation procedure (techniques of asymptotics beyond all orders) for obtaining the estimate of the amplitude of the oscillatory tails associated with the weakly non-local solitary wave solutions of the fifth-order KdV equation was devised by Grimshaw and Joshi [21] and Pomeau et al. [28] (see also [14]) following the related work by Kruskal and Segur [27], Segur and Kruskal [30] and Segur et al. [31]. They showed that the amplitude of the oscillation is exponentially small that lies beyond all orders of the usual long wave expansion. Akylas and Yang [2] using the method of Fourier transform and a perturbation analysis in the wave number domain estimated this exponentially small amplitude of the tail oscillations. In this method, the amplitude of the far-field oscillations is determined easily without the need for asymptotic matching in the complex plane, as required in the techniques of asymptotics beyond all orders. Boyd [15] computed these weakly non-local solitary waves numerically using the Newton–Kantorovich pseudospectral (collocation) method based on the rational Chebyshev and radiation basis functions. The numerical estimate of the amplitude of tail oscillations were compared with the analytical estimate obtained by Pomeau et al. [28]. He has called these weakly non-local solitary waves as ‘nanopterons’ and also drawn attention to their prevalence in a variety of physical systems. A detailed account of various asymptotic methods and their applications to weakly non-local solitary waves and related topics can be found in a recent review article by Boyd [13].

In this paper, we provide detailed analytical and numerical studies of the singularly perturbed (sixth-order) Boussinesq equation

$$\eta_{tt} = \eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx} + \epsilon^2 \eta_{xxxxx}, \quad (1.1)$$

where ϵ is a small parameter. This equation was originally introduced by Daripa and Hua [18] as a regularization of the classical (illposed) Boussinesq equation which corresponds to $\epsilon = 0$ in Eq. (1.1).

The classical Boussinesq equation possesses solitary wave solutions. However, as an initial value problem (IVP), it suffers from severe short wave instability. The linearized version of this equation admits solutions of the form $e^{\sigma t + ikx}$ with short wave instability $\sigma \approx k^2$ as $k \rightarrow \infty$. A consequence of this short wave instability is possible non-existence of classical solutions to this equation for arbitrary initial data except for some isolated solutions such as the classical

solitary wave solutions. Another consequence of this short wave instability is difficulty in constructing good approximate solutions of even known solutions [17,18]. These facts seriously cast doubts on the real utility of this Boussinesq equation in spite of its frequent appearance in most books on non-linear waves and water waves (e.g., [24,36]) as a model equation for bi-directional propagation of small amplitude long-waves.

Daripa and Hua [18] attempted to compute the approximate solutions of the illposed Boussinesq equation using the regularized sixth-order Boussinesq equation (1.1) subject to initial data of solitary wave type. However, for small ϵ , their computations resulted in solutions which behave like solitary waves at their cores and oscillations of small amplitude at their tails. So, their computed solutions had the behavior of weakly non-local solitary wave solutions of the fifth-order KdV equation [2,15,21,22,28], and full non-linear water wave equations for $0 < \tau < 1/3$ [9,32–34].

In this paper, we construct the weakly non-local solitary wave solutions of the regularized sixth-order (singularly perturbed) Boussinesq equation (1.1) in the form of traveling waves by using various analytical and numerical methods originally devised to obtain this type of weakly non-local solitary wave solutions of the fifth-order (singularly perturbed) KdV equation. We also obtain the estimate of the amplitude of the oscillatory tails associated with these weakly non-local solitary waves. The layout of the paper is as follows.

In Section 2, we address the physical relevance of the sixth-order (singularly perturbed) Boussinesq equation (1.1) in the context of water waves. In particular, we show that this equation actually describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for Bond number τ less than but very close to $1/3$. In Section 3, we use a regular perturbation analysis to find an approximate traveling wave solution of this equation in the core region. Also on the basis of far-field analyses and heuristic arguments, we show that, unlike the classical solitary waves, the traveling wave solutions of this equation do not vanish to zero at infinity. Instead, they possess small amplitude fast oscillations at infinity. In Section 4, using the technique of asymptotics beyond all orders [27,30,31], we estimate this exponentially small amplitude of the tail oscillations by following closely the approach of Grimshaw and Joshi [21] and Pomeau et al. [28]. In Section 5, we use the method of Fourier transform coupled with a perturbation analysis in the wave number domain as in [2] to obtain an estimate of this exponentially small amplitude of the far-field oscillations. In Section 6, we compute the weakly non-local solitary waves numerically using the Newton–Kantorovich pseudospectral (collocation) method based on the rational Chebyshev and radiation basis functions as in [15]. The analytical and numerical results are presented and compared in Section 7. The discussions and concluding remarks are given in Section 8.

2. Physical relevance of the sixth-order Boussinesq equation

Before focusing on the nature of traveling wave solutions of the sixth-order Boussinesq equation, we briefly comment on the physical relevance of this equation in the context of water waves. Below we show that this equation actually describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for Bond number τ less than but very close to $1/3$ (i.e., $\tau \uparrow 1/3$). In particular, we present a formal derivation of this equation from two-dimensional potential flow equations for water waves in the limit $\tau \uparrow 1/3$ through an asymptotic series expansion for small amplitude and long wavelength as in [24,36].

Let $h(x, t) = h_0 + a\eta(x, t)$ represent the free water surface (where h_0 is the height of undisturbed water surface, $\eta(x, t)$ is the surface wave showing the mean disturbance on the undisturbed water surface, and a is the amplitude of the surface wave), $z = 0$ represent the bottom topography, and ϕ denote the potential function. Under scaling

$$x \rightarrow lx, \quad z \rightarrow h_0z, \quad t \rightarrow \frac{l}{\sqrt{gh_0}}t, \quad \phi \rightarrow \frac{la\sqrt{gh_0}}{h_0}\phi, \quad (2.1)$$

where l is the wavelength of surface wave and g is the acceleration due to gravity, the governing equation and boundary conditions for water waves (see [24,36]) are given by

$$\beta\phi_{xx} + \phi_{zz} = 0, \quad (2.2)$$

with

$$\phi_z = 0 \quad \text{at } z = 0, \quad (2.3a)$$

$$\eta_t + \alpha\eta_x\phi_x - \frac{1}{\beta}\phi_z = 0 \quad \text{at } z = 1 + \alpha\eta, \quad (2.3b)$$

$$\phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_z^2 + \eta - \beta\tau\frac{\eta_{xx}}{[1 + \alpha^2\beta\eta_x^2]^{3/2}} = 0 \quad \text{at } z = 1 + \alpha\eta, \quad (2.3c)$$

where $\alpha = a/h_0$ is the amplitude parameter, $\beta = (h_0/l)^2$ is the wavelength parameter, and $\tau = \Gamma/\rho gh_0^2$ is the surface tension parameter or the Bond number. Here Γ is the surface tension coefficient.

The linearized version of the above equation admits solutions of the form $Ae^{ikx - i\omega t}$ provided the following dispersion relation holds (see [36]).

$$\omega^2 = \frac{c_0^2}{h_0^2} [(1 + \tau k^2 h_0^2)kh_0 \tanh(kh_0)], \quad (2.4)$$

where $c_0 = \sqrt{gh_0}$. In the long wavelength limit ($\beta \ll 1$, i.e., $kh_0 \ll 1$), we have

$$\omega^2 = c_0^2 k^2 \left[1 - \left(\frac{1}{3} - \tau \right) k^2 h_0^2 + \frac{1}{3} \left(\frac{2}{5} - \tau \right) k^4 h_0^4 - \frac{2}{15} \left(\frac{17}{42} - \tau \right) k^6 h_0^6 + \frac{17}{315} \left(\frac{62}{153} - \tau \right) k^8 h_0^8 - \dots \right]. \quad (2.5)$$

This indicates that the leading order dispersion term in the equation for η is of the order $((1/3) - \tau)k^2 h_0^2$, or equivalently of the order $((1/3) - \tau)\beta$. Therefore, the leading order dispersion term is $O(\beta)$ if $\beta \ll ((1/3) - \tau) \leq (1/3)$, i.e., $((1/3) - \tau) = O(1)$. On the other hand, the leading order dispersion term is $O(\beta^2)$ if $((1/3) - \tau) = O(\beta)$ which will be true when $\tau \uparrow 1/3$. However, the non-linear term is always of the order α , irrespective of the value of the Bond number τ . A balance between non-linear and dispersive effects (which is necessary to model a solitary wave) requires that $\alpha = O(\beta)$ when $((1/3) - \tau) \gg \beta$, and $\alpha = O(\beta^2)$ when $((1/3) - \tau) = O(\beta)$.

We now derive the necessary sixth-order Boussinesq equation for η from Eqs. (2.2) and (2.3a)–(2.3c) by suitably eliminating ϕ from these equations under the limiting conditions

$$\left(\frac{1}{3} - \tau \right) = K_1 \beta \quad \text{and} \quad \alpha = K_2 \beta^2 \quad \text{as } \beta \rightarrow 0, \quad (2.6)$$

with non-zero constants K_1 and K_2 are fixed. In doing so, we seek a solution for the potential function ϕ in the form (see [24,36])

$$\phi = \sum_{k=0}^{\infty} (-1)^k \beta^k \frac{z^{2k}}{(2k)!} \frac{\partial^{2k} f}{\partial x^{2k}}, \quad (2.7)$$

where $f = f(x, t)$ is the value of potential function ϕ at $z = 0$. Expression (2.7) suggests the fact that the horizontal velocity ϕ_x is of $O(1)$, where as, the vertical velocity ϕ_z is of $O(\beta)$. The potential function ϕ satisfies the Laplace equation (2.2) and the bottom boundary condition (2.3a). Substituting expansion (2.7) in the free surface conditions (2.3b) and (2.3c) and rearranging the series, we obtain

$$\eta_t + \alpha \eta_x \sum_{k=0}^{\infty} (-1)^k \beta^k \frac{(1 + \alpha \eta)^{2k}}{(2k)!} \frac{\partial^{2k+1} f}{\partial x^{2k+1}} + \sum_{k=0}^{\infty} (-1)^k \beta^k \frac{(1 + \alpha \eta)^{2k+1}}{(2k+1)!} \frac{\partial^{2k+2} f}{\partial x^{2k+2}} = 0 \quad (2.8)$$

and

$$\sum_{k=0}^{\infty} (-1)^k \beta^k \frac{(1 + \alpha\eta)^{2k}}{(2k)!} \frac{\partial^{2k+1} f}{\partial t \partial x^{2k}} + \frac{\alpha}{2} \left[\sum_{k=0}^{\infty} (-1)^k \beta^k \frac{(1 + \alpha\eta)^{2k}}{(2k)!} \frac{\partial^{2k+1} f}{\partial x^{2k+1}} \right]^2 + \frac{\alpha\beta}{2} \left[\sum_{k=0}^{\infty} (-1)^k \beta^k \frac{(1 + \alpha\eta)^{2k+1}}{(2k+1)!} \frac{\partial^{2k+2} f}{\partial x^{2k+2}} \right]^2 + \eta - \beta\tau\eta_{xx} [1 + \alpha^2\beta\eta_x^2]^{-3/2} = 0. \tag{2.9}$$

Substituting Eq. (2.6) in Eqs. (2.8) and (2.9), and retaining terms only up to $O(\beta^2)$, we obtain

$$\eta_t + K_2\beta^2\eta_x f_x + (1 + K_2\beta^2\eta) f_{xx} - \frac{\beta}{6} f_{xxx} + \frac{\beta^2}{120} f_{xxxxx} + O(\beta^3) = 0 \tag{2.10}$$

and

$$f_t - \frac{\beta}{2} f_{xxt} + \frac{\beta^2}{24} f_{xxxxt} + \frac{K_2\beta^2}{2} f_x^2 + \eta - \frac{\beta}{3} \eta_{xx} + K_1\beta^2\eta_{xx} + O(\beta^3) = 0. \tag{2.11}$$

Differentiating Eq. (2.11) with respect to x and letting $u = f_x$ (horizontal velocity at the bottom $z = 0$), we have system (2.10) and (2.11) in the form

$$\eta_t + u_x - \frac{\beta}{6} u_{xxx} + K_2\beta^2(\eta u)_x + \frac{\beta^2}{120} u_{xxxxx} + O(\beta^3) = 0 \tag{2.12}$$

and

$$\eta_x + u_t - \frac{\beta}{3} \eta_{xxx} - \frac{\beta}{2} u_{xxt} + K_1\beta^2\eta_{xxx} + K_2\beta^2uu_x + \frac{\beta^2}{24} u_{xxxxt} + O(\beta^3) = 0. \tag{2.13}$$

From Eqs. (2.12) and (2.13), we see that

$$\eta_t + u_x = O(\beta) \quad \text{and} \quad \eta_x + u_t = O(\beta) \tag{2.14}$$

and

$$\eta_t + u_x - \frac{\beta}{6} u_{xxx} = O(\beta^2) \quad \text{and} \quad \eta_x + u_t - \frac{\beta}{3} \eta_{xxx} - \frac{\beta}{2} u_{xxt} = O(\beta^2). \tag{2.15}$$

After obtaining $(\partial/\partial t)(\text{Eq. (2.12)}) - (\partial/\partial x)(\text{Eq. (2.13)})$ and then using the lower-order approximations (2.14) and (2.15) to alter the higher-order derivative terms, we find

$$\eta_{tt} - \eta_{xx} - K_2\beta^2 \left[\frac{1}{2} \eta^2 + u^2 \right]_{xx} - K_1\beta^2\eta_{xxx} - \frac{\beta^2}{45} \eta_{xxxxx} + O(\beta^3) = 0. \tag{2.16}$$

We need to eliminate u from Eq. (2.16). From Eq. (2.14a), we have

$$u = - \int_{-\infty}^x \eta_t dx + O(\beta). \quad (2.17)$$

Using Eq. (2.17), we have Eq. (2.16) in the form

$$\begin{aligned} \eta_{tt} - \eta_{xx} - K_2 \beta^2 \left[\frac{1}{2} \eta^2 + \left(\int_{-\infty}^x \eta_t dx \right)^2 \right]_{xx} - K_1 \beta^2 \eta_{xxxx} - \frac{\beta^2}{45} \eta_{xxxxxx} \\ = 0, \end{aligned} \quad (2.18)$$

where we have dropped $O(\beta^3)$ from the equation. This equation is one version of the sixth-order Boussinesq equation appropriate for the approximate description of the bi-directional propagating small amplitude long capillary-gravity waves on the surface of shallow water for $\tau \uparrow 1/3$. Another version will be introduced below. At first sight, Eq. (2.18) looks rather complicated. But if we use the co-ordinate transformation

$$X = \frac{1}{\sqrt{K_1}} \left(x + K_2 \beta^2 \int_{-\infty}^x \eta(x, t) dx \right), \quad T = \frac{1}{\sqrt{K_1}} t \quad (2.19)$$

and substitute

$$N = \frac{3K_2}{2} (\eta - K_2 \beta^2 \eta^2), \quad (2.20)$$

then Eq. (2.18) becomes

$$N_{TT} - N_{XX} - \beta^2 (N^2)_{XX} - \beta^2 N_{XXX} - \epsilon^2 \beta^2 N_{XXXX} = 0, \quad (2.21)$$

where $\epsilon^2 = (1/45K_1^2)$. From Eq. (2.6), it is to be noted that $\tau \uparrow 1/3$ can hold true in the limit $K_1 \rightarrow \infty$ and $\beta \rightarrow 0$. Moreover, we see that ϵ^2 can be considered a small parameter independent of the wavelength parameter β .

Since the parameter associated with the sixth-order term in Eq. (2.21) is relatively small compared to the other terms, it will be worthwhile to study this equation for small values of the parameter ϵ with $\beta = 1$. Eq. (1.1) can be recovered from Eq. (2.21) if we write (η, x, t) instead of (N, X, T) with $\beta = 1$.

2.1. Connection with the fifth-order KdV equation

The fifth-order KdV equation [22] can be derived from the sixth-order Boussinesq equation (2.21) by using the following far-field co-ordinate transformations:

$$\xi = X - T \quad \text{and} \quad \tau = \beta^2 T. \quad (2.22)$$

The transformation (2.22) describes a wave which changes slowly in a reference frame moving with velocity one (the non-dimensional shallow water velocity).

The leading order terms in the transformed equation correspond to the following fifth-order KdV equation

$$N_\tau + NN_\xi + \frac{1}{2}N_{\xi\xi\xi} + \frac{1}{90K_1^2}N_{\xi\xi\xi\xi\xi} = 0. \tag{2.23}$$

If we further use the change of variables

$$\xi \rightarrow \frac{\xi}{\sqrt{2\delta}}, \quad \tau \rightarrow \frac{\tau}{\delta\sqrt{2\delta}}, \quad N \rightarrow \delta N, \tag{2.24}$$

then Eq. (2.23) reduces to the desired canonical form:

$$N_\tau + NN_\xi + N_{\xi\xi\xi} + \epsilon^2 N_{\xi\xi\xi\xi\xi} = 0, \tag{2.25}$$

which appears as Eq. (2.29) in Hunter and Scheurle [22]. Here $\epsilon^2 = 2\delta/45K_1^2$.

3. Analyses of the problem

Since Eq. (1.1) has solitary wave solutions for $\epsilon = 0$, the natural question arises whether Eq. (1.1) also admits solitary wave solutions for small values of ϵ . Therefore, we seek a traveling wave solution of equation (1.1) in the form

$$\eta(x, t) = \eta(x - ct), \tag{3.1}$$

where c is the phase speed (velocity) of the wave. Substitution of Eq. (3.1) in Eq. (1.1) and using x for the new variable $x - ct$ yields

$$(1 - c^2)\eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx} + \epsilon^2\eta_{xxxxx} = 0. \tag{3.2}$$

The question now becomes whether Eq. (3.2) admits solutions which decay exponentially to zero as $x \rightarrow \pm\infty$ for any small positive value of ϵ . Since we are interested in bounded solutions of Eq. (3.2) as $x \rightarrow \pm\infty$, on integrating Eq. (3.2) twice and taking the constants of integration as zero, we obtain

$$(1 - c^2)\eta + \eta^2 + \eta_{xx} + \epsilon^2\eta_{xxxx} = 0. \tag{3.3}$$

3.1. Core analysis: regular perturbation method

We seek the solution of Eq. (3.3) in the form of a regular asymptotic expansion

$$\begin{aligned} \eta &= \eta_0 + \epsilon^2\eta_1 + \dots, \\ c &= c_0 + \epsilon^2c_1 + \dots \end{aligned} \tag{3.4}$$

Substitution of expansions (3.4) in Eq. (3.3) yields the following $O(\epsilon^0)$ and $O(\epsilon^2)$ equations:

$$(1 - c_0^2)\eta_0 + \eta_0^2 + \eta_{0xx} = 0 \quad (3.5)$$

and

$$(1 - c_0^2)\eta_1 - 2c_0c_1\eta_0 + 2\eta_0\eta_1 + \eta_{1xx} + \eta_{0xxx} = 0. \quad (3.6)$$

The solitary wave solution of Eq. (3.5) is given by

$$\eta_0 = 6\gamma^2 \operatorname{sech}^2(\gamma x), \quad (3.7)$$

where γ is related to c_0 by

$$c_0 = \pm \sqrt{1 + 4\gamma^2}. \quad (3.8)$$

γ is a free parameter characterizing the width of the solitary wave (3.7). For the solution of Eq. (3.6) to exist, we must have the following compatibility condition:

$$-2c_0c_1 = (1 - c_0^2)^2, \quad (3.9)$$

which gives

$$c_1 = \pm \frac{8\gamma^4}{\sqrt{1 + 4\gamma^2}}. \quad (3.10)$$

The solution of Eq. (3.6) is then given by

$$\eta_1 = -10\gamma^2\eta_0 + \frac{5}{2}\eta_0^2. \quad (3.11)$$

Thus, the solutions of η and c are given by

$$\eta = (1 - 10\gamma^2\epsilon^2 + \dots)\eta_0 + \left(\frac{5}{2}\epsilon^2 + \dots\right)\eta_0^2 + \dots, \quad (3.12)$$

and

$$c = \pm \left[\sqrt{1 + 4\gamma^2} + \epsilon^2 \frac{8\gamma^4}{\sqrt{1 + 4\gamma^2}} + \dots \right]. \quad (3.13)$$

If we use γ to characterize the core solitary wave, then Eq. (3.13) provides an approximate relation between the phase speed c and γ . From Eq. (3.13) we see that

$$c^2 - 1 \approx 4\gamma^2 + \epsilon^2(4\gamma^2)^2. \quad (3.14)$$

It is to be noted here that expansion (3.4) can be continued to arbitrarily higher order. The general term η_n in Eq. (3.4) will be an $(n + 1)$ th order polynomial in η_0 . Since η_0 is symmetric about $x = 0$ and decays down to zero exponentially at tail ends (i.e., as $x \rightarrow \pm\infty$), the form of solution (3.12) implies that η will also be symmetric about $x = 0$ and will decay down to zero exponentially as

$x \rightarrow \pm\infty$. So, by the method of regular asymptotic analysis, we only get exponentially decaying solution in the far-field. However, as we will see below in Section 3.2, the far-field analysis contradicts this.

3.2. Far-field analysis: heuristic arguments

If we assume that η is small in the far-field $x \rightarrow \pm\infty$, then Eq. (3.3) linearizes to

$$(1 - c^2)\eta + \eta_{xx} + \epsilon^2\eta_{xxxx} = 0 \quad \text{as } x \rightarrow \pm\infty. \tag{3.15}$$

Eq. (3.15) has solutions of the form $\eta = \exp(ipx)$ provided

$$\epsilon^2 p^4 - p^2 = (c^2 - 1). \tag{3.16}$$

Since $c > 1$, Eq. (3.16) has two real roots (which correspond to the oscillatory behavior of η at infinity) and two purely imaginary roots (which correspond to decaying and growing behavior of η at infinity). For a local solitary wave, only the root which corresponds to the decaying behavior of η at infinity is acceptable. This then implies the necessity of three independent boundary conditions on η as $x \rightarrow \infty$, with three more as $x \rightarrow -\infty$, leading altogether to the necessity of six independent boundary conditions on η for a fourth-order differential equation (3.15). Therefore, we cannot force η to vanish at both $x \rightarrow \infty$ and $x \rightarrow -\infty$. There will be an oscillatory behavior at least on one side at infinity.

The real (p_r) and imaginary (p_i) roots of Eq. (3.16) are, respectively, given by

$$\begin{aligned} p_r^2 &= \frac{1 + [1 + 4\epsilon^2(c^2 - 1)]^{1/2}}{2\epsilon^2} \\ &= \frac{1}{\epsilon^2} + (c^2 - 1) - \epsilon^2(c^2 - 1)^2 + \dots \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} p_i^2 &= \frac{1 - [1 + 4\epsilon^2(c^2 - 1)]^{1/2}}{2\epsilon^2} \\ &= -(c^2 - 1) + \epsilon^2(c^2 - 1)^2 - \frac{2}{3}\epsilon^4(c^2 - 1)^3 + \dots \end{aligned} \tag{3.18}$$

Here we note that the real roots are large and the imaginary roots are finite for small ϵ . If we use γ to characterize the wave, we must express p_r and p_i in terms of γ . Substituting the expression for $c^2 - 1$ given by Eq. (3.12) in Eqs. (3.17) and (3.18), we obtain

$$p_r^2 \approx \frac{1}{\epsilon^2} + 4\gamma^2 \tag{3.19}$$

and

$$p_1^2 \approx -4\gamma^2. \quad (3.20)$$

Therefore, for η to be bounded, it must be of the form

$$\eta = A_{1\pm} \cos\left(\frac{q}{\epsilon}x\right) + A_{2\pm} \sin\left(\frac{q}{\epsilon}x\right) \quad \text{as } x \rightarrow \pm\infty, \quad (3.21)$$

where $A_{1\pm}$ and $A_{2\pm}$ are some ϵ -dependent unknown constants and $q = |p_r|\epsilon$. From Eq. (3.17) or (3.19), it follows that $q \rightarrow 1$ as $\epsilon \rightarrow 0$. So, the frequency of oscillations $|p_r| = (q/\epsilon) \rightarrow (1/\epsilon)$ as $\epsilon \rightarrow 0$, and hence, the far-field oscillations are very fast. It is clear from Eq. (3.21) that, in general, there will be oscillatory behaviors on both the sides at infinity. Also, the amplitude of oscillations on either ends may be different. By combining the sine and cosine terms via trigonometric identities, we can write Eq. (3.21) in the form

$$\eta = A_{\pm} \sin\left(\frac{q}{\epsilon}(x + \phi_{\pm})\right) \quad \text{as } x \rightarrow \pm\infty. \quad (3.22)$$

Here A_{\pm} and ϕ_{\pm} are, respectively, the amplitude and phase shift constant of the oscillatory tails as $x \rightarrow \pm\infty$. For symmetric weakly non-local solitary wave solutions, $A_+ = A_- = A$ and $\phi_+ = \phi_- = \phi$. In Sections 4–6, we will obtain estimates of the amplitude A of the tail oscillations through various analytical and numerical methods.

Note: If we change the sign of the sixth-order derivative term from ‘positive’ to ‘negative’ in the singularly perturbed (sixth-order) Eq. (1.1), then the characteristic equation (i.e. the analog of Eq. (3.16) for the corresponding linearized fourth-order ordinary differential equation) in the far-field becomes

$$\epsilon^2 p^4 + p^2 = -(c^2 - 1). \quad (3.23)$$

All four roots of Eq. (3.24) are purely imaginary. They are

$$\begin{aligned} p_{1,2}^2 &= \frac{-1 + [1 - 4\epsilon^2(c^2 - 1)]^{1/2}}{2\epsilon^2} \\ &= -\left[(c^2 - 1) + \epsilon^2(c^2 - 1)^2 + \frac{2}{3}\epsilon^4(c^2 - 1)^3 + \dots\right] \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} p_{3,4}^2 &= \frac{-1 - [1 - 4\epsilon^2(c^2 - 1)]^{1/2}}{2\epsilon^2} \\ &= -\frac{1}{\epsilon^2} + [(c^2 - 1) + \epsilon^2(c^2 - 1)^2 + \dots]. \end{aligned} \quad (3.25)$$

We see that, two roots will cause the solutions to be unbounded in the far-field, while the other two will cause the solutions to vanish there. This is true for each of the far-field. Therefore, only four far-field boundary conditions

(two on each side) are required in order to have local solitary wave solutions of the fourth-order equation (3.3) with fourth-order term negative which is feasible.

4. Perturbation analysis in the complex plane

In this section, we will construct a solution of Eq. (3.3) which behaves like the solution (3.12) in the core region and solution (3.22) in the far-field by using the technique of asymptotics beyond all orders [21,27,28,30,31]. We will also show that the amplitude of the far-field oscillations is exponentially small that lies beyond all orders in the regular asymptotic expansion of form (3.4) or (3.12).

Since $\eta_0(x)$ given by Eq. (3.7) is singular in the complex x -plane at $x = \pm(2n + 1)i\pi/2\gamma$ ($n = 0, 1, 2, \dots$), the core solution $\eta(x)$ given by Eq. (3.12) cannot describe the actual behavior of the solution of Eq. (3.3) in the neighborhood of these singular points. In fact, the perturbation term $\epsilon^2\eta_{xxxx}$ cannot be considered as of lower order than the other terms in Eq. (3.3) in the neighborhood of these singular points. So, it is important to consider the solution structure of Eq. (3.3) near these singular points. To do this, we need to consider a rescaling through which the small parameter ϵ^2 is removed from the highest derivative term in Eq. (3.3). This problem is called the *inner problem*.

We consider the singularity closest to the real axis in the upper half-plane. We introduce the following inner variables y and η_i :

$$x = \frac{i\pi}{2\gamma} + \epsilon y \quad \text{and} \quad \eta_i = \epsilon^2 \eta. \tag{4.1}$$

The subscript i refers to the inner problem. When Eq. (4.1) is substituted in Eq. (3.3), we obtain

$$\epsilon^2(1 - c^2)\eta_i + \eta_i^2 + \eta_{iyy} + \eta_{iyyyy} = 0. \tag{4.2}$$

Now neglecting the term containing the small parameter ϵ^2 , we have the inner problem as

$$\eta_i^2 + \eta_{iyy} + \eta_{iyyyy} = 0. \tag{4.3}$$

To find the solution of the original problem (3.3), we need to solve the inner problem (4.3) and connect the asymptotic behavior of the inner solution at large distances to that of the core (outer) solution (3.12) by matching their asymptotic behaviors in a region where they both make sense.

To the leading order, the asymptotic behavior of η_0 near the singularity $i\pi/2\gamma$ is given by

$$\eta_0 = -6\gamma^2 \operatorname{cosech}^2(\gamma\epsilon y) \approx -\frac{6}{(\epsilon y)^2} \quad \text{as } \epsilon y \rightarrow 0. \quad (4.4)$$

Therefore, to the leading order, the asymptotic behavior of the outer solution (3.12) near the singularity $i\pi/2\gamma$ will be given by

$$\eta \approx \frac{1}{\epsilon^2} \left(-\frac{6}{y^2} + \frac{90}{y^4} \right) \quad \text{as } \epsilon y \rightarrow 0. \quad (4.5)$$

Hence, it should be matched to a solution of the inner problem with the asymptotic behavior

$$\eta_i \approx -\frac{6}{y^2} + \frac{90}{y^4} \quad \text{as } |y| \rightarrow \infty \quad (\text{or } |y| \gg 1). \quad (4.6)$$

In view of Eq. (4.6), the solution of the inner problem (4.3) is constructed as an asymptotic series in $1/y^2$ of the form

$$\eta_i \approx -\frac{6}{y^2} + \frac{90}{y^4} + \sum_{n=3}^{\infty} \frac{a_n}{y^{2n}} \quad \text{as } |y| \rightarrow \infty \quad (\text{or } |y| \gg 1). \quad (4.7)$$

When Eq. (4.7) is substituted into Eq. (4.3), the coefficients of $y^{-(2n+4)}$ give

$$\begin{aligned} & (2n-2)(2n-1)(2n)(2n+1)a_{n-1} + (2n+4)(2n-3)a_n + \sum_{k=2}^{n-1} a_k a_{n+1-k} \\ & = 0 \quad \text{for } n \geq 3, \end{aligned} \quad (4.8)$$

with $a_1 = -6$ and $a_2 = 90$. So, a_n 's can be obtained from Eq. (4.8) recursively. As $n \rightarrow \infty$, the non-linear term in Eq. (4.8) becomes less important. Therefore, an asymptotic formula for a_n correct up to $O(1/n^2)$ is given by

$$\begin{aligned} & (2n-2)(2n-1)(2n)(2n+1)a_{n-1} + (2n+4)(2n-3)a_n \\ & \approx 0 \quad \text{for large } n. \end{aligned} \quad (4.9)$$

Eq. (4.9) recursively gives

$$a_n \approx \frac{(2n+1)(2n-1)}{(2n+2)(2n+4)} (-1)^n (2n-1)! K \quad \text{for large } n, \quad (4.10)$$

where K is some constant. The value of K is obtained by computing the exact values of a_n from Eq. (4.8) for some large values of n and matching it with the asymptotic formula (4.10). The value of K was found to be 59.91.

With the coefficients a_n 's given by Eq. (4.8) for all $n \geq 3$ and by Eq. (4.9) or (4.10) for large n , the asymptotic series solution (4.7) of the inner problem (4.3) diverges for all y . However, it can still be summed using the method of Borel summation [10]. So, we express $\eta_i(y)$ in the form of a Laplace transform (see also [28,21] given by

$$\eta_i(y) = \int_0^\infty V\left(\frac{p}{y}\right) e^{-p} dp, \tag{4.11}$$

where $V(s)$ is an unknown function. The integration path in Eq. (4.11) extends from 0 to ∞ in the half-plane $\text{Re } p > 0$. We can also rewrite the expression (4.11) for $\eta_i(y)$ in the forms

$$\eta_i(y) = y \int_0^\infty V(s) e^{-sy} ds = \int_0^\infty V'(s) e^{-sy} ds, \tag{4.12}$$

where, without loss of any generality, we have assumed that $V(0) = 0$. $V'(s)$ denotes the derivative of $V(s)$ w.r.t. s . The integration path in Eq. (4.12) extends from 0 to ∞ in the half-plane $\text{Re}(sy) > 0$. The first identity in Eq. (4.12) is obtained from Eq. (4.11) by substituting $p = sy$. The second identity follows from the first through integration by parts (or, more directly through the properties of Laplace transform).

We can find the unknown function $V(s)$ or $V'(s)$ by substituting the asymptotic series (4.7) in Eq. (4.12) and taking the inverse Laplace transform which yields

$$V(s) = \sum_{n=1}^\infty b_n s^{2n} \quad \text{and} \quad V'(s) = \sum_{n=1}^\infty 2nb_n s^{2n-1}, \tag{4.13}$$

where

$$b_n = \frac{a_n}{(2n)!} \approx \frac{(2n+1)(2n-1)}{(2n+2)(2n+4)} (-1)^n \frac{K}{2n} \quad \text{for large } n. \tag{4.14}$$

It is readily established that the series (4.13) for $V(s)$ and $V'(s)$ converges for $|s| < 1$ (so the radius of convergence is unity) and has a singularity at $s = \pm i$. However, the singularity of $V(s)$ and $V'(s)$ at $s = \pm i$ and the non-linear term in the ordinary differential equation (4.3) would imply that $V(s)$ and $V'(s)$ will also have singularity at $s = \pm 2i, \pm 3i, \dots$, so on. If $y = -iY, Y \in \mathbb{R}_+$, then the integrand $V(p/y)$ becomes singular at $p = \pm kY, k = 1, 2, \dots$. The singularities at $p = +kY$ lie exactly on the integration path in Eq. (4.11), and therefore, it has to be deformed clockwise to avoid the singularity, as shown in Fig. 1.

We now study the behavior of $V(s)$ in the neighborhood of the singularity at $s = +ki, k = 1, 2, \dots$. Since $b_n \approx (-1)^n K/2n$ as $n \rightarrow \infty$ (by Eq. (4.14)), we see that $V(s)$ behaves like $K \ln(1 + i(s/k))$ in the neighborhood of the singularity at $s = +ki$. Therefore, we have

$$V \approx K \ln \left(1 + i \frac{p}{ky} \right) \quad \text{as } \frac{p}{y} \rightarrow +ki. \tag{4.15}$$

If $y = -iY, Y \in \mathbb{R}_+$, and $p \rightarrow kY_-$, then the value of the above logarithm will be real, and we will have Eq. (4.15) in the form

$$V \approx K \ln \left(1 - \frac{p}{kY} \right) \quad \text{as } p \rightarrow kY_-. \tag{4.16}$$

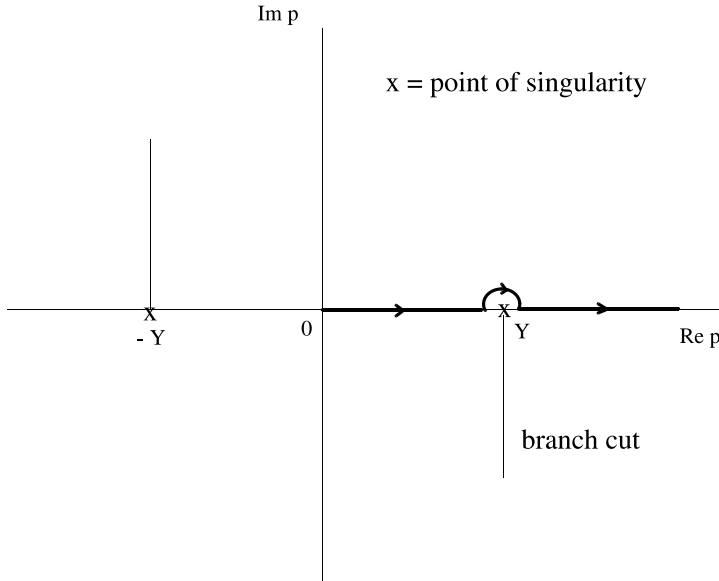


Fig. 1. Deformation of the integration path around the branch cut of the singularity at $p = Y$ in a clockwise direction. The point of singularity $p = Y$ lies on the real axis in $\text{Re } p > 0$.

But, if $y = -iY, Y \in \mathbb{R}_+$ and $p \rightarrow kY_+$, then the value of the above logarithm will be complex, and since we deform the integration path in clockwise direction near the singularity, we will have equation (4.15) in the form

$$V \approx K \left[\ln \left(\frac{p}{kY} - 1 \right) - i\pi \right] \quad \text{as } p \rightarrow kY_+. \tag{4.17}$$

Therefore, when y is purely imaginary and negative (i.e., $y = -iY, Y \in \mathbb{R}_+$), the integrand in Eq. (4.11), in the neighborhood of the singularity at $p = +kY, k = 1, 2, \dots$, is obtained as

$$\begin{aligned}
 V \left(\frac{p}{y} \right) e^{-p} &= V \left(\frac{ip}{Y} \right) e^{-p} \\
 &\approx \begin{cases} K \ln \left(1 - \frac{p}{kY} \right) e^{-p} & \text{for } p = kY_-, \\ K \ln \left(\frac{p}{kY} - 1 \right) e^{-p} - i\pi K e^{-p} & \text{for } p = kY_+. \end{cases} \tag{4.18}
 \end{aligned}$$

Therefore, from Eq. (4.11), we have $\eta_i(y)$ as

$$\eta_i(y) = \eta_i(-iY) \approx PV \int_0^\infty V \left(\frac{ip}{Y} \right) e^{-p} dp - i\pi K \sum_{k=1}^\infty e^{-kY}. \tag{4.19}$$

The integral in Eq. (4.19) is the Cauchy principal value (PV) integral which excludes the contributions from the singularities at $p = kY$, $k = 1, 2, \dots$. The leading contribution from the singularities comes from the singularity at $p = Y$ which is equal to $-i\pi K e^{-Y}$. For large Y (i.e., $|y| \gg 1$), the Cauchy principal value integral must agree with the asymptotic series (4.7) with $y = -iY$, and hence, we obtain

$$\eta_i(y) = \eta_i(-iY) \approx \sum_{n=1}^{\infty} (-1)^n \frac{a_n}{Y^{2n}} - i\pi K \sum_{k=1}^{\infty} e^{-kY}. \tag{4.20}$$

It is clear from Eq. (4.20) that, an exponentially small correction in the inner solution appears in the asymptotic series of the inner solution beyond all orders. Therefore, there should be a corresponding exponentially small correction in the outer solution which will appear in the algebraic asymptotic series of the outer solution beyond all order. When we match the inner solution (4.20) to the outer solution, we obtain the solution of Eq. (3.3) as

$$\eta(x) \approx \eta_0(x) + \epsilon^2 \eta_1(x) + \dots - \frac{i\pi K}{\epsilon^2} \sum_{k=1}^{\infty} \exp \left[-k \left(\frac{\pi}{2\gamma\epsilon} + i \frac{x}{\epsilon} \right) \right]. \tag{4.21}$$

When x is purely real, $\eta(x)$ should be real. Therefore, the correct matching will lead to

$$\eta(x) \approx \eta_0(x) + \epsilon^2 \eta_1(x) + \dots + \frac{\pi K}{\epsilon^2} \sum_{k=1}^{\infty} e^{-(k\pi/2\gamma\epsilon)} \sin \left(\frac{k|x|}{\epsilon} \right). \tag{4.22}$$

Therefore, the symmetric traveling wave solution of the singularly perturbed (sixth-order) Boussinesq equation (1.1) has small amplitude oscillatory behavior at its tail ends which is explicitly given by

$$\eta(x) \approx \frac{\pi K}{\epsilon^2} \sum_{k=1}^{\infty} e^{-(k\pi/2\gamma\epsilon)} \sin \left(\frac{k|x|}{\epsilon} \right) \quad \text{as } x \rightarrow \mp\infty. \tag{4.23}$$

The dominant term in the above sum is $(\pi K/\epsilon^2)e^{-(\pi/2\gamma\epsilon)} \sin(|x|/\epsilon)$. It is worth pointing out that the frequency of oscillation of the oscillatory tails is of $O(1/\epsilon)$ which is same as predicted in the far-field analysis of Section 3.2.

5. Perturbation analysis in the Fourier domain

In this section, we will construct the oscillatory tails and estimate their amplitudes by using the method of Fourier transform coupled with a perturbation analysis in the Fourier domain as in [2].

The form of solution (3.12) implies that $\eta \rightarrow 0$ as $x \rightarrow \pm\infty$. So, taking the Fourier transform of Eq. (3.12), we obtain

$$\hat{\eta} = (1 - 10\gamma^2\epsilon^2 + \dots)\hat{\eta}_0 + (5\epsilon^2/2 + \dots)\hat{\eta}_0^2 + \dots, \quad (5.1)$$

where the Fourier transform of $\eta(x)$ is defined by

$$\hat{\eta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(x)e^{-ikx} dx. \quad (5.2)$$

From Eq. (3.7), the Fourier transform of η_0 is given by

$$\begin{aligned} \hat{\eta}_0 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 6\gamma^2 \operatorname{sech}^2(\gamma x) e^{-ikx} dx \\ &= 3k \operatorname{cosech}(\pi k/2\gamma). \end{aligned} \quad (5.3)$$

By the convolution theorem, we have the Fourier transform of η_0^2 as

$$\begin{aligned} \hat{\eta}_0^2 &= \int_{-\infty}^{\infty} \hat{\eta}_0(l)\hat{\eta}_0(k-l) dl \\ &= 9 \int_{-\infty}^{\infty} l(k-l) \operatorname{cosech}(\pi l/2\gamma) \operatorname{cosech}(\pi(k-l)/2\gamma) dl \\ &= 3k(k^2 + 4\gamma^2) \operatorname{cosech}(\pi k/2\gamma). \end{aligned} \quad (5.4)$$

Thus, by the help of Eqs. (5.3) and (5.4), we have Eq. (5.1) in the form

$$\begin{aligned} \hat{\eta} &= (3k + 15k^3\epsilon^2/2 + \dots) \operatorname{cosech}(\pi k/2\gamma) \\ &= \frac{1}{\epsilon} f(\tilde{k}) \operatorname{cosech}(\pi \tilde{k}/2\epsilon\gamma), \end{aligned} \quad (5.5)$$

where

$$\tilde{k} = k\epsilon \quad \text{and} \quad f(\tilde{k}) = 3\tilde{k} + 15\tilde{k}^2/2 + \dots \quad (5.6)$$

Now taking the Fourier transform of the differential equation (3.3), we obtain

$$[(1 - c^2) - k^2 + \epsilon^2 k^4] \hat{\eta}(k) + \int_{-\infty}^{\infty} \hat{\eta}(l)\hat{\eta}(k-l) dl = 0. \quad (5.7)$$

By the help of Eqs. (5.5) and (5.6), Eq. (5.7) becomes

$$\begin{aligned} &[\epsilon^2(1 - c^2) - \tilde{k}^2 + \tilde{k}^4] f(\tilde{k}) \\ &+ \sinh\left(\frac{\pi \tilde{k}}{2\gamma\epsilon}\right) \int_{-\infty}^{\infty} \frac{f(\tilde{l})f(\tilde{k} - \tilde{l}) d\tilde{l}}{\sinh(\pi \tilde{l}/2\gamma\epsilon) \sinh(\pi(\tilde{k} - \tilde{l})/2\gamma\epsilon)}. \end{aligned} \quad (5.8)$$

It can be easily shown that

$$\begin{aligned} & \sinh\left(\frac{\pi\tilde{k}}{2\gamma\epsilon}\right) \int_{-\infty}^{\infty} \frac{f(\tilde{l})f(\tilde{k}-\tilde{l})d\tilde{l}}{\sinh(\pi\tilde{l}/2\gamma\epsilon)\sinh(\pi(\tilde{k}-\tilde{l})/2\gamma\epsilon)} \\ & = 2 \int_0^{\tilde{k}} f(\tilde{l})f(\tilde{k}-\tilde{l})d\tilde{l} \quad \text{as } \epsilon \rightarrow 0. \end{aligned} \tag{5.9}$$

Thus, to the leading order in ϵ , Eq. (5.8) can be approximated to the following Volterra integral equation for $f(\tilde{k})$:

$$\tilde{k}^2(\tilde{k}^2 - 1)f(\tilde{k}) + 2 \int_0^{\tilde{k}} f(\tilde{l})f(\tilde{k} - \tilde{l})d\tilde{l} = 0. \tag{5.10}$$

We express the solution of $f(\tilde{k})$ as a power series

$$f(\tilde{k}) = \sum_{m=0}^{\infty} b_m \tilde{k}^{2m+1}. \tag{5.11}$$

Comparing Eq. (5.11) with Eq. (5.6), we get $b_0 = 3$ and $b_1 = 15/2$. Now we need to obtain b_m for $m \geq 2$. Substituting Eq. (5.11) in Eq. (5.10), we obtain

$$\sum_{m=0}^{\infty} b_m \tilde{k}^{2m+5} - \sum_{m=0}^{\infty} b_m \tilde{k}^{2m+3} + 2 \sum_{m=0}^{\infty} \sum_{r=0}^m b_r b_{m-r} \int_0^{\tilde{k}} \tilde{l}^{2r+1} (\tilde{k} - \tilde{l})^{2m-2r+1} d\tilde{l} = 0. \tag{5.12}$$

It can be easily shown that

$$\int_0^{\tilde{k}} \tilde{l}^{2r+1} (\tilde{k} - \tilde{l})^{2m-2r+1} d\tilde{l} = \frac{(2m - 2r + 1)!(2r + 1)!}{(2m + 3)!} \tilde{k}^{2m+3}. \tag{5.13}$$

With the help of Eq. (5.13), (5.12) becomes

$$\sum_{m=1}^{\infty} b_{m-1} \tilde{k}^{2m+3} - \sum_{m=0}^{\infty} b_m \tilde{k}^{2m+3} + 2 \sum_{m=0}^{\infty} \sum_{r=0}^m \frac{(2m - 2r + 1)!(2r + 1)!}{(2m + 3)!} b_r b_{m-r} \tilde{k}^{2m+3} = 0. \tag{5.14}$$

Equating the coefficients of \tilde{k}^{2m+3} to zero, we obtain $b_0 = 3, b_1 = 15/2$, and

$$\begin{aligned} & - \left[\frac{(2m - 1)(2m + 6)}{(2m + 3)(2m + 2)} \right] b_m + b_{m-1} \\ & + 2 \sum_{r=1}^{m-1} \frac{(2m - 2r + 1)!(2r + 1)!}{(2m + 3)!} b_r b_{m-r} = 0, \quad m \geq 2. \end{aligned} \tag{5.15}$$

As $m \rightarrow \infty$, the non-linear term in Eq. (5.15) becomes less important. So, we obtain

$$b_m \approx b_{m-1} \approx C \quad \text{as } m \rightarrow \infty, \quad (5.16)$$

where C is a constant. The value of C can be obtained by evaluating the values of b_m from Eq. (5.15) up to some large values of m . The value of C is found to be 29.96. So, $C = K/2$, where $K = 59.91$, as obtained in Section 4. Thus, the series (5.11) for f will be convergent for $|\tilde{k}| < 1$ and will have pole singularities at $\tilde{k} = \pm 1$. Therefore, we will have

$$\begin{aligned} f(\tilde{k}) &\approx \frac{K\tilde{k}}{2(1-\tilde{k}^2)} \quad \text{as } \tilde{k} \rightarrow \pm 1 \\ &\approx -\frac{K}{4(\tilde{k} \mp 1)} \quad \text{as } \tilde{k} \rightarrow \pm 1. \end{aligned} \quad (5.17)$$

Then in view of Eqs. (5.5) and (5.17), $\hat{\eta}$ will have pole singularities at $\tilde{k} = \pm 1$, and we will have

$$\begin{aligned} \hat{\eta} &\approx -\frac{K}{4\epsilon(\tilde{k} \mp 1)} \operatorname{cosech}(\pi\tilde{k}/2\gamma\epsilon) \quad \text{as } \tilde{k} \rightarrow \pm 1 \\ &\approx -\frac{K}{4\epsilon^2(k \mp 1/\epsilon)} \operatorname{cosech}(\pi k/2\gamma) \quad \text{as } k \rightarrow \pm 1/\epsilon \\ &\approx \mp \frac{K}{2\epsilon^2(k \mp 1/\epsilon)} e^{-\pi/2\gamma\epsilon} \quad \text{as } k \rightarrow \pm 1/\epsilon, \end{aligned} \quad (5.18)$$

where we have used the following asymptotic relations:

$$\operatorname{cosech}(\pi k/2\gamma) \approx \begin{cases} 2e^{-\pi/2\gamma\epsilon} & \text{as } k \rightarrow 1/\epsilon, \\ -2e^{-\pi/2\gamma\epsilon} & \text{as } k \rightarrow -1/\epsilon. \end{cases} \quad (5.19)$$

Taking the inverse transform of $\hat{\eta}(k)$, we have

$$\eta(x) = \text{PV} \int_{-\infty}^{\infty} \hat{\eta}(k) e^{ikx} dk + \int_{C_{-1/\epsilon}} \hat{\eta}(k) e^{ikx} dk + \int_{C_{1/\epsilon}} \hat{\eta}(k) e^{ikx} dk, \quad (5.20)$$

where $C_{-1/\epsilon}$ and $C_{1/\epsilon}$ represent the integration path (half circles) near the singularity at $-1/\epsilon$ and $1/\epsilon$, respectively, as shown in Fig. 2. The first integral in Eq. (5.20) is the Cauchy principal value integral which must agree with the asymptotic expansion (3.4) or (3.12). By the residue theorem, we have Eq. (5.20) in the form

$$\eta(x) \approx \eta_0(x) + \epsilon^2 \eta_1(x) + \dots - \frac{i\pi K}{2\epsilon^2} \exp \left[-\left(\frac{\pi}{2\gamma\epsilon} \right) \right] \left(\exp \left[-i\frac{x}{\epsilon} \right] - \exp \left[i\frac{x}{\epsilon} \right] \right). \quad (5.21)$$

Eq. (5.21) is the required non-local solitary wave solution of the sixth-order (singularly perturbed) Boussinesq equation (1.1). Thus, we have the far-field oscillation in the form

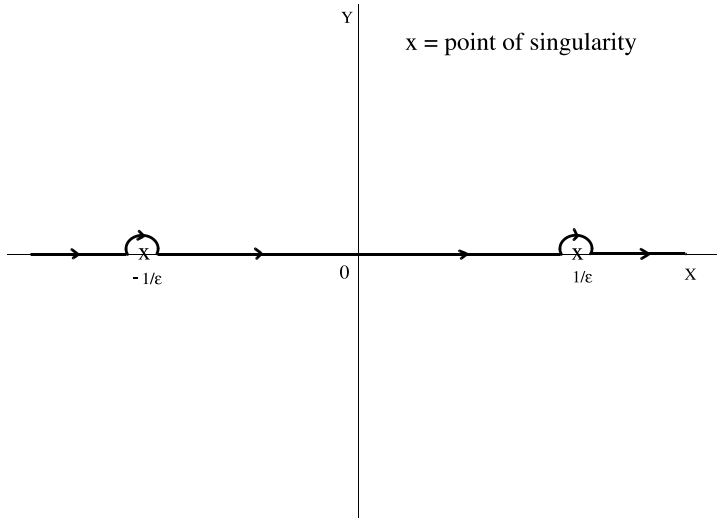


Fig. 2. Deformation of the integration path around the singularities at $k = -1/\epsilon$ and $k = 1/\epsilon$ in a clockwise direction.

$$\eta(x) \approx \frac{\pi K}{\epsilon^2} \exp \left[- \left(\frac{\pi}{2\gamma\epsilon} \right) \right] \sin \left(\frac{|x|}{\epsilon} \right) \quad \text{as } x \rightarrow \mp\infty. \tag{5.22}$$

This estimate agrees with the estimate (4.23) to leading order.

6. Newton–Kantorovich equation and pseudospectral method

In this section, we will obtain a numerical solution for the reduced traveling wave ordinary differential equation (3.3) of the singularly perturbed (sixth-order) Boussinesq equation (1.1) which will behave like the core solitary wave solution (3.12) near $x = 0$ and far-field oscillatory solution (3.22) as $x \rightarrow \pm\infty$. For this, we will derive the Newton–Kantorovich equation for the differential equation (3.3) and describe a pseudospectral (collocation) method to solve this equation iteratively by using a combination of rational Chebyshev and radiation basis function. Since the method is described in detail in Boyd [15], we only present an outline of the method here.

Since Eq. (3.3) is non-linear, it is solved iteratively. Suppose $\eta^{(i)}(x)$ is the solution at i th iterate and $\delta\eta^{(i)}(x)$ is a correction to $\eta^{(i)}(x)$ such that

$$\eta(x) = \eta^{(i)}(x) + \delta\eta^{(i)}(x), \tag{6.1}$$

satisfies Eq. (3.3). Substituting Eq. (6.1) in Eq. (3.3) and linearizing the LHS, we get the following linear inhomogeneous ODE (known as Newton–Kantorovich equation) for the iterative scheme

$$\begin{aligned} & ((1 - c^2) + 2\eta^{(i)})\delta\eta^{(i)} + \delta\eta_{xx}^{(i)} + \epsilon^2\delta\eta_{xxxx}^{(i)} \\ & = -[(1 - c^2) + \eta^{(i)}]\eta^{(i)} + \eta_{xx}^{(i)} + \epsilon^2\eta_{xxxx}^{(i)}. \end{aligned} \quad (6.2)$$

The Newton–Kantorovich iteration procedure is repeated until the correction $\delta\eta^{(i)}(x)$, or equivalently, the RHS of Eq. (6.2) becomes negligibly small. The iteration procedure requires an initial guess. For small values ϵ , the core solitary wave solution (3.12) is taken as the initial guess which depends on the phase speed c through its dependence on the core solitary wave width parameter γ . For a given value of c , γ is obtained as the positive real root of Eq. (3.14). Eq. (3.3) can also be solved for larger values of ϵ . In this case, the method of continuation (see [15]) is used to find a suitable initial guess, since use of solution (3.12) as an initial guess may not yield a convergent solution. In continuation method, we start finding the solution of Eq. (3.3) for a small value of ϵ , as described above. Then for an increased value of ϵ , the converged solution for the previous value of ϵ is used as the initial guess.

We now describe the pseudospectral (collocation) method to solve the Newton–Kantorovich equation (6.2). The spectral basis functions for the pseudospectral method are chosen suitably a combination of rational Chebyshev and radiation basis function to get correct core solitary wave and far-field oscillatory behaviors. So, following Boyd [15], if we write the solution at i th iterate as

$$\eta^{(i)}(x) = \sum_{n=1}^{N-1} a_n^{(i)} \Phi_n(x) + \Phi_{\text{rad}}(x; A^{(i)}), \quad (6.3)$$

then the correction to the solution at i th iterate will be given by

$$\delta\eta^{(i)}(x) \approx \sum_{n=1}^{N-1} \delta a_n^{(i)} \Phi_n(x) + \delta A^{(i)} \Phi_{\text{rad},A}(x; A^{(i)}). \quad (6.4)$$

Here, A is the amplitude of the tail oscillations which is obtained as a part of the solution along with the spectral coefficients a_n , $n = 1, 2, \dots, N - 1$. The spectral basis functions $\Phi_n(x)$, $n = 1, 2, \dots, N - 1$ and $\Phi_{\text{rad}}(x; A)$ are constructed as follows (see also [15])

$$\begin{aligned} \Phi_n(x) &= \text{TB}_{2n}(x) - 1 \\ &= \cos[2n \cot^{-1}(x/L)] - 1, \quad L = 2/\gamma \end{aligned} \quad (6.5)$$

and

$$\Phi_{\text{rad}}(x; A) = H(x)\eta_{\text{cn}}(x; A) + H(-x)\eta_{\text{cn}}(-x; A). \quad (6.6)$$

Since the rational Chebyshev functions $\text{TB}_{2n}(x)$ are even and asymptote to 1 as $x \rightarrow \pm\infty$, the basis functions $\Phi_n(x)$ are even and decay down to zero at tail ends. Thus, the series $\sum_{n=1}^{N-1} a_n \Phi_n(x)$ gives the right behavior of the symmetric

core solitary wave with peak at $x = 0$. The oscillatory behavior of the solution at tail ends is visualised by the radiation basis function $\Phi_{\text{rad}}(x; A)$ through its dependence on cnoidal function $\eta_{\text{cn}}(x, A)$ and smoothed step function $H(x)$ which are discussed below.

The cnoidal function $\eta_{\text{cn}}(x, A)$ agrees with the form of the far-field solution (3.22) to the leading order in A . Therefore, it describes the far-field more accurately. Also, it is another approximate solution of the differential equation (3.3) for all x , not only in the far-field $x \rightarrow \pm\infty$ (see [15] for more discussion). Using Stokes series approximations and cnoidal matching, and following Boyd [15], we obtain the corresponding cnoidal function for the differential equation (3.3) as

$$\eta_{\text{cn}}(x; A) = A \sin\left(\frac{q}{\epsilon}(x + \phi)\right) + A^2 \left[C_1 + C_2 \cos\left(\frac{2q}{\epsilon}(x + \phi)\right) \right] + A^3 C_3 \sin\left(\frac{3q}{\epsilon}(x + \phi)\right) + O(A^4) \tag{6.7}$$

and

$$q = q_0 + A^2 q_2 + O(A^4), \tag{6.8}$$

where

$$\begin{aligned} q_0 &= (1 + 4\epsilon^2 \gamma^2)^{1/2}, & q_2 &= \frac{\epsilon^4(C_2 - 2C_1)}{2q_0^3 - q_0}, \\ C_1 &= \frac{\epsilon^2}{2(q_0^4 - q_0^2)}, & C_2 &= \frac{\epsilon^2}{(30q_0^4 - 6q_0^2)}, \\ C_3 &= \frac{\epsilon^4}{48(50q_0^8 - 15q_0^4 + q_0^2)}. \end{aligned} \tag{6.9}$$

The phase shift constant $\phi = 0$ corresponds to the case in which both the core solitary wave and the oscillatory tails are in phase.

The smoothed step function $H(x)$ is suitably chosen in order to have the asymptotic behavior $H(x) \sim 1$ as $x \rightarrow \infty$ and $H(x) \sim 0$ as $x \rightarrow -\infty$. For simplicity, as in [15], we choose

$$H(x) = \frac{1}{2} [1 + \tanh(\gamma(x + \phi))]. \tag{6.10}$$

Since we are interested in obtaining symmetric non-local solitary wave solution of Eq. (3.3) with peak at $x = 0$, we choose the N spectral grid (collocation) points all on positive real axis given by

$$x_n = L \cot[(2n - 1)\pi/4N], \quad n = 1, 2, \dots, N. \tag{6.11}$$

At i th iterate, $\eta^{(i)}$, $A^{(i)}$ and $a_n^{(i)}$ s are known. We need to compute the corresponding corrections $\delta A^{(i)}$ and $\delta a_n^{(i)}$ s from the Newton–Kantorovich equation (6.2). Substituting the spectral series (6.4) into the Newton–Kantorovich equation (6.2) and demanding that the residual vanish at N collocation points defined above, we obtain the matrix equation $JE = F$, where $E = [\delta a_1^{(i)}, \delta a_2^{(i)}, \dots, \delta a_{N-1}^{(i)}, \delta A^{(i)}]^T$, $F = [F_1^{(i)}, F_2^{(i)}, \dots, F_N^{(i)}]^T$ and $J = [J_{nj}^{(i)}]$ is the Jacobian matrix of the resulting system of equations. Explicitly $J_{nj}^{(i)}$ and $F_n^{(i)}$ for $n = 1, 2, \dots, N$ are expressed as

$$J_{nj}^{(i)} = \begin{cases} [((1 - c^2) + 2\eta^{(i)})\phi_j + \phi_{j,xx} + \epsilon^2\phi_{j,xxxx}]|_{x=x_n} \\ \text{for } j = 1, 2, \dots, N - 1, \\ \\ [((1 - c^2) + 2\eta^{(i)})\phi_{\text{rad},A} + \phi_{\text{rad},Axx} + \epsilon^2\phi_{\text{rad},Axxxx}]|_{x=x_n} \\ \text{for } j = N, \end{cases} \quad (6.12)$$

and

$$F_n^{(i)} = [((1 - c^2) + \eta^{(i)})\eta^{(i)} + \eta_{xx}^{(i)} + \epsilon^2\eta_{xxxx}^{(i)}]|_{x=x_n}. \quad (6.13)$$

The various derivatives of the basis functions involved in the calculation of Jacobian matrix J through Eq. (6.12) and RHS column vector F through Eq. (6.13) can be obtained explicitly. The matrix equation $JE = F$ is solved for $\delta a_1^{(i)}, \delta a_2^{(i)}, \dots, \delta a_{N-1}^{(i)}, \delta A^{(i)}$ using a direct numerical method such as Gaussian elimination with partial pivoting. Then the spectral coefficients are corrected through $a_n^{(i+1)} = a_n^{(i)} + \delta a_n^{(i)}$, $n = 1, 2, \dots, N - 1$ and $A^{(i+1)} = A^{(i)} + \delta A^{(i)}$. Then the new solution, new Jacobian matrix and new RHS vector are evaluated using the updated values $a_n^{(i+1)}$'s and $A^{(i+1)}$. Then the matrix equation is solved again. The iteration procedure is continued until the maximum (L_∞) norm of the vector E , or equivalently, F becomes negligibly small. Also, the final/converged spectral coefficients are stored to consider as the initial guess for computing solution for the next higher value of the perturbation parameter ϵ .

7. Numerical results

The amplitude A of the oscillatory tails obtained through the analytical estimate (4.23) or (5.22) for different values of the perturbation parameter ϵ^2 and phase speed c is shown in Table 1. It is observed that, the amplitude A of the oscillatory tails is exponentially small as compared to the amplitude of the core which is approximately equal to $6\gamma^2$ or $1.5(c^2 - 1)$. Also it decreases exponentially fast as the value of ϵ and c decreases.

The numerical results are obtained for phase shift constant $\phi = 0$ and various values of the perturbation parameter ϵ^2 and phase speed c . However, the results are presented with respect to a combined (group) parameter $\epsilon^2(c^2 - 1)$. The

Table 1

Analytical estimate of the amplitude A of the oscillatory tails for different values of the perturbation parameter ϵ^2 and phase speed c

$\epsilon^2 \setminus c$	1.05	1.10	1.15	1.20	1.25
0.0025	0.430405E-80	0.206454E-54	0.640770E-43	0.520739E-36	0.293631E-31
0.0100	0.433390E-38	0.295349E-25	0.162456E-19	0.458033E-16	0.107685E-13
0.0225	0.301413E-24	0.106437E-15	0.704676E-12	0.138968E-09	0.523737E-08
0.0400	0.209484E-17	0.529839E-11	0.383407E-08	0.199358E-06	0.300032E-05
0.0625	0.238900E-13	0.310892E-08	0.594467E-06	0.138618E-04	0.120067E-03
0.0900	0.112281E-10	0.201547E-06	0.158387E-04	0.216081E-03	0.129365E-02
0.1225	0.861493E-09	0.374824E-05	0.155859E-03	0.144863E-02	0.665788E-02
0.1600	0.214244E-07	0.321447E-04	0.828458E-03	0.577118E-02	0.217482E-01
0.2025	0.252448E-06	0.165261E-03	0.293416E-02	0.163294E-01	0.527304E-01
0.2500	0.176862E-05	0.595707E-03	0.784714E-02	0.364899E-01	0.104153E+00

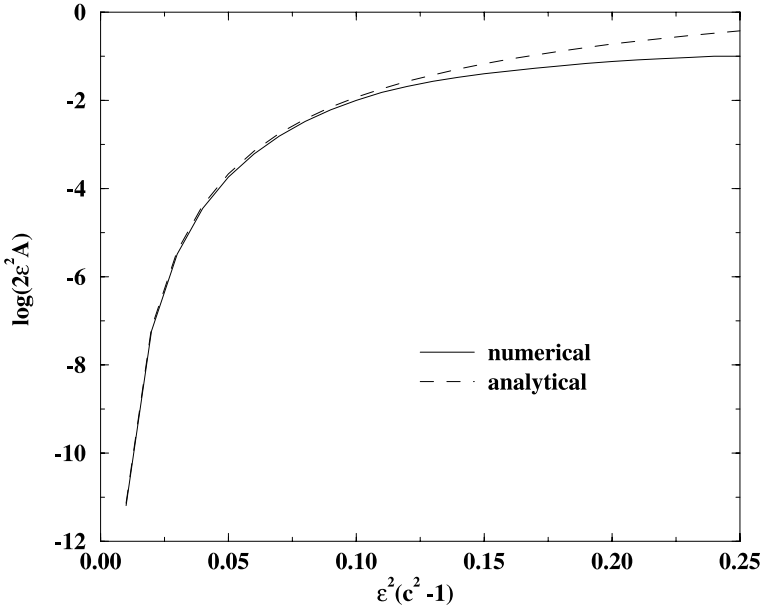


Fig. 3. Comparison of the numerically computed amplitude (solid lines) of the oscillatory tails with that of the analytical estimate (dashed lines).

numerically computed amplitude of the oscillatory tails is compared with the corresponding analytical estimate in Fig. 3. This figure shows the variation of $2\epsilon^2 A$ with the group parameter $\epsilon^2(c^2 - 1)$. It is observed that the numerically computed amplitude of the far-field oscillations agrees well with the analytical estimate for small values of $\epsilon^2(c^2 - 1)$. However, for larger values of $\epsilon^2(c^2 - 1)$, there is a small discrepancy between the two estimates which is expected since

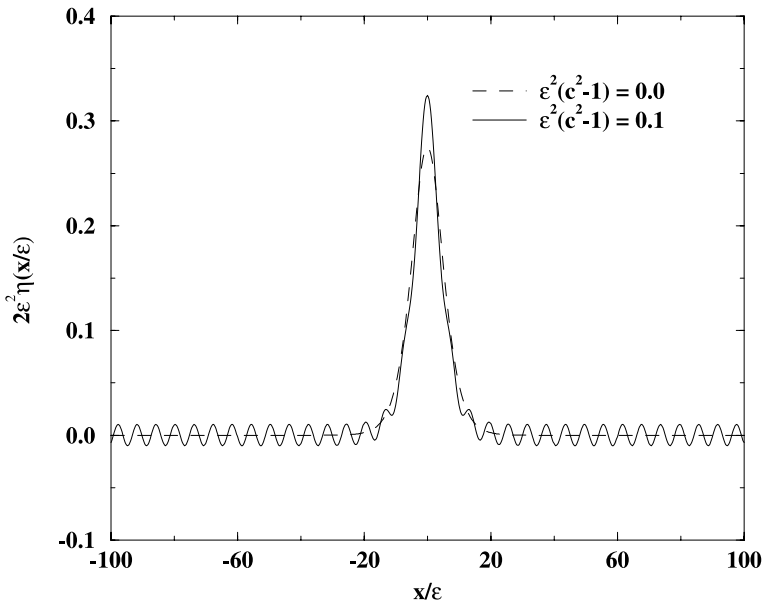


Fig. 4. The plots for travelling wave solutions of the singularly perturbed (sixth-order) Boussinesq equation for $\epsilon^2(c^2 - 1) = 0.1$ and $\epsilon^2(c^2 - 1) = 0.0$.

the analytical estimate is based on the asymptotic analysis for $\epsilon \ll 1$. Also, it is to be noted that the amplitude decreases exponentially fast as the value of $\epsilon^2(c^2 - 1)$ decreases.

Fig. 4 shows the numerically computed symmetric weakly non-local solitary wave solution of the sixth-order (singularly perturbed) Boussinesq equation (1.1) for $\epsilon^2(c^2 - 1) = 0.1$. For this moderate value of $\epsilon^2(c^2 - 1)$, the oscillatory tail is clearly visible. However, the oscillatory tail is very (exponentially) small in comparison to the amplitude of the core solitary wave which is centered on the origin $x = 0$. The core in the neighborhood of $x = 0$ is best described by the solution (3.12). As the value of $\epsilon^2(c^2 - 1)$ decreases, the oscillatory tails decrease and collapse almost into the local solitary wave solution of the classical Boussinesq equation, as seen in the left-hand side graph of Fig. 5. The oscillatory tails are there, but are so small that they are invisible in comparison to the peak of the wave. However, if we zoom near the tails, the oscillations are clearly visible as seen in the right-hand side graph of Fig. 5.

8. Discussions and concluding remarks

In Daripa and Hua [18], a singularly perturbed (sixth-order) Boussinesq equation was introduced as a dispersive regularization of the ill-posed classical

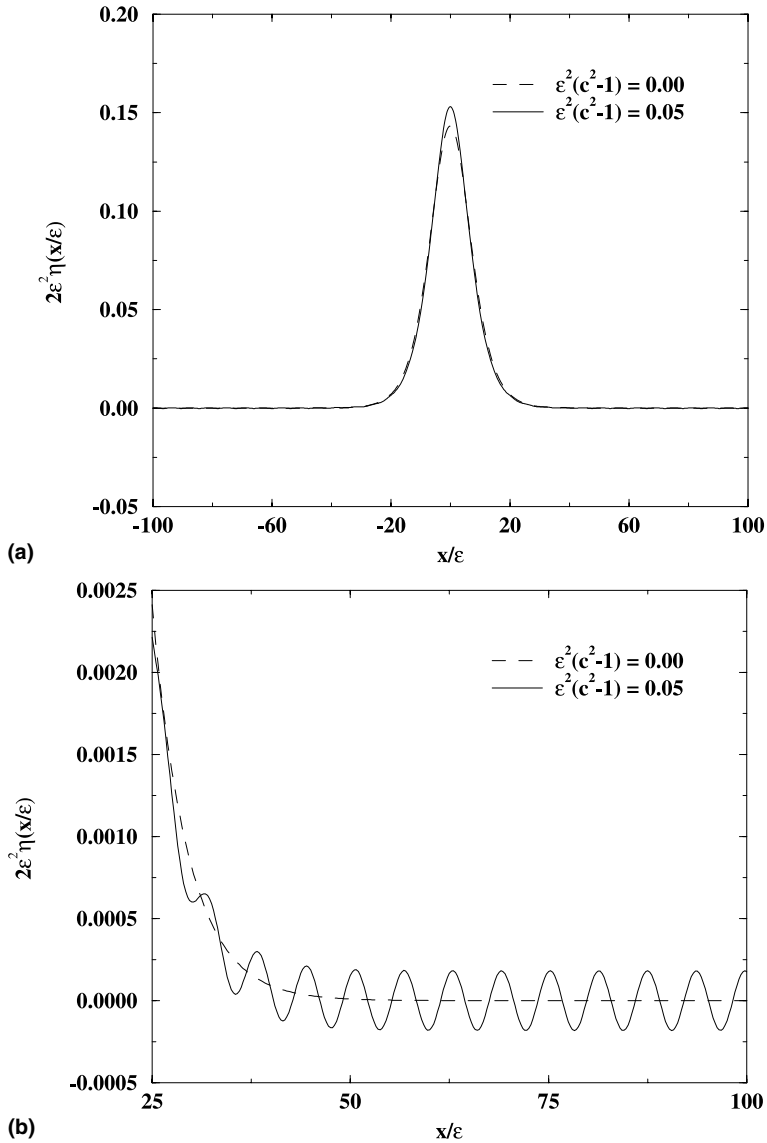


Fig. 5. The plots for travelling wave solutions of the singularly perturbed (sixth-order) Boussinesq equation for $\epsilon^2(c^2-1)=0.05$ and $\epsilon^2(c^2-1)=0.0$. The left-hand side graph of the figure shows the full plot, where as, the right-hand side graph of the figure shows the zoomed plot near the oscillatory tail.

(fourth-order) Boussinesq equation. In this paper, we showed the physical relevance of this equation in the context of water waves. In particular, we derived this equation from two-dimensional potential flow equations governing

the shallow water waves under gravity using an asymptotic series expansion in the limit of small amplitude (i.e., $\alpha \ll 1$) and long wavelength (i.e., $\beta \ll 1$) with $\alpha = O(\beta^2)$ as $\beta \rightarrow 0$ and Bond number τ less than but very close to $1/3$ (i.e., $\tau \uparrow 1/3$). Thus, this equation is valid up to $O(\beta^2)$, meaning that it can serve as a better model than the classical fourth-order (illposed) Boussinesq equation to describe bi-directional wave propagation on the surface of shallow water. Also it is established that the singularly perturbed (fifth-order) KdV equation derived by Hunter and Scheurle [22] can be recovered from this singularly perturbed (sixth-order) Boussinesq equation by using a suitable far-field co-ordinate transformation as discussed in Section 2.1.

Motivated by the numerical work of Daripa and Hua [18] where they obtained weakly non-local solitary wave solutions of the regularized sixth-order Boussinesq equation subject to the initial data of solitary wave type, we analyzed this equation to find the traveling wave solutions. On the basis of far-field analyses and heuristic arguments, we established that, unlike the classical solitary waves, the traveling wave solutions of this regularized sixth-order Boussinesq equation cannot vanish in the far-field. Instead, such waves must possess small amplitude fast oscillations at distances far from the core of the waves extending up to infinity. This behavior confirms the numerical prediction of Daripa and Hua [18]. So, the traveling wave solutions of this equation have the behavior of the weakly non-local solitary wave solutions of the singularly perturbed (fifth-order) KdV equation [2,15,21,22,28]), and the full non-linear water wave equations for $0 < \tau < 1/3$ [9,32–34].

We reviewed various analytical (see [2,21,28]) and numerical (see [15]) methods originally devised to obtain this type of weakly non-local solitary wave solutions of the fifth-order (singularly perturbed) KdV equation. Using these methods, we obtain the weakly non-local solitary wave solutions of the regularized sixth-order (singularly perturbed) Boussinesq equations and provide the estimate of the amplitude of oscillations which persist far from the core solitary wave. The analytical estimate of the amplitude agreed with that of the numerical estimate for small values of the perturbation parameter ϵ . Also, although the analytical estimate of the tail oscillations is similar to that obtained by Akylas and Yang [2], Grimshaw and Joshi [21] and Pomeau et al. [28] for the fifth-order KdV equation, the estimate in the present case is different from their estimates because of the different estimate of the constant K and different relation between the phase speed c of the wave and the core solitary wave width parameter γ .

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