Appl. Math. Lett. Vol. 4, No. 1, pp. 91–94, 1991
Printed in Great Britain.

0893-9659/91 $3.00 + 0.00

Using the Grid Spacing Ratio As a Continuous Variable in One Dimensional Adaptive Grid Generation

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(Received June 1990)

Abstract. A new method for one dimensional adaptive grid generation is introduced based on defining the grid spacing ratio as a continuous variable. In this paper we validate our theoretical results in order to justify their use in numerical construction of adaptive grids in one dimension.

1. INTRODUCTORY REVIEW

The use of adaptive grids is very common in solving ordinary or partial differential equations which exhibit frontal or boundary layer types of behavior. The motivation behind using adaptive grids is to improve the accuracy of solutions at an optimal computational cost by concentrating grid points in appropriate areas. There are many methods which have been developed to generate such adaptive grids [1,2,3] in one and two dimensions. In this paper we briefly describe a new method of generating such adaptive grids in one dimension (see also [4]). This method is based on approximating the grid spacing ratio as a continuous variable and using this as the prime generator of these adaptive grids. This method ensures that the grid spacing ratio is a smooth function and provides very high resolution. Oscillations in the grid spacing ratio are not desirable as these may cause serious problems in obtaining accurate solutions of partial differential equations numerically. This is particularly true when implicit methods are used since the grid spacing ratio affects the matrix which needs to be inverted either directly or iteratively. Most often this property is overlooked while generating these adaptive grids. In our method, possibilities of such oscillations do not arise. Additionally, in our method an adaptive grid ratio can be generated in a direct manner so that the grid spacing ratio is within some specified limits. This will be addressed in a forthcoming paper [5]. We now describe the basic ideas behind our method and some numerical results. The ideas presented here and the potential application of this method in the light of the above discussion justifies this brief note. An elaborate account of this method from numerical and theoretical viewpoints is addressed elsewhere [5].

2. THE NUMERICAL METHOD

Let Ω_ζ and Ω_x be intervals in ζ and x space respectively. Consider a one dimensional monotonic mapping $x(\zeta) : \Omega_\zeta \to \Omega_x$, with the property that uniformly spaced ζ-grid points (i.e., the grid points on ζ axis) are mapped to nonuniformly spaced x-grid points. The mapping $x(\zeta)$ will be constructed so that rapidly varying functions $f(x)$ map into slowly varying functions $f(\zeta)$. The desired adaptivity is incorporated in the choice of $x(\zeta)$. Let us denote by h_ζ the spacing between two consecutive ζ-grid points and $x_i = x(\zeta_i)$. The x-grid points constitute an adaptive grid.

The grid spacing ratio at the grid point x_i is defined as the ratio of two adjacent grid spacings, i.e.,

$$r_i = \frac{x_{i+1} - x_i}{x_i - x_{i-1}}. \tag{1}$$

This research was partially supported by the NSF grant DMS-8803669.

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The grid spacing ratio is required to be positive for a monotonic mapping. This definition is extended as a continuous variable in the following manner. The equation (1) can be rewritten as

$$\frac{r_i+1}{r_i-1}=\frac{x_{i+1}-x_{i-1}}{x_{i+1}-2x_i+x_{i-1}}. \tag{2}$$

Using a Taylor series approximation, equation (2) can be written as

$$\frac{r(x)+1}{r(x)-1}=\frac{2}{h_\zeta}\frac{x_\zeta+O(h_\zeta^2)}{x_{\zeta\zeta}+O(h_\zeta^2)}. \tag{3}$$

In (3) we have suppressed the subscript 'i'. If we introduce the notation,

$$\alpha=h_\zeta\frac{x_{\zeta\zeta}}{x_\zeta} \tag{4}$$

in (3), then it can be reduced to

$$r(x)=\frac{2+\alpha}{2-\alpha}+O(h_\zeta^2). \tag{5}$$

For numerical purposes, we approximate (5) by dropping the second and higher order terms in (5). By using the definition (1) of the grid spacing ratio, the grid construction equation can then be written down as

$$\frac{x_{i+1}-x_i}{x_i-x_{i-1}}=\frac{2+\alpha(x_i)}{2-\alpha(x_i)}. \tag{6}$$

This can be rewritten in a convenient form for numerical purposes:

$$x_{i+1}=x_i+(x_i-x_{i-1})*\frac{2+\alpha(x_i)}{2-\alpha(x_i)}. \tag{7}$$

The dependency of α on the function $f(x)$ to be adapted can be written down explicitly if one desires. However, we derive its dependency through

$$x_\zeta=cw(x;k) \text{ on } x_l\le x\le x_u, \tag{8}$$

which is most commonly used for mapping $\Omega_\zeta\rightarrow\Omega_x$. In (8), '$c$' is a constant, k is an adjustable parameter and $w(x;k)$ depends on k and the function $f(x)$. The adaptive grids are generated on the x axis. Without loss of generality, the indices of the adaptive grid points on the x axis are taken as the coordinates of the corresponding ζ-grid points, i.e., $\Omega_\zeta:[\zeta_0=0,\zeta_N=N]\rightarrow\Omega_x:[x_0=x_l,x_N=x_u]$. Thus the uniform spacing is $h_\zeta=1$. This convention gives a domain Ω_ζ that is dynamically increasing as the number of x-grid points increases. Thus the second order accurate approximation of $r(x)$, eqn. (6), should be interpreted as $O(\frac{1}{N^2})$ accurate, which is the order of the derivatives hidden in the $O(h_\zeta^2)$ terms in (5).

¿From (4) and (8), it follows easily that

$$\alpha(x;k)=cw_x(x;k). \tag{9}$$

The constant 'c' in (9) is obtained by direct integration of (8) from $x_0=x_l$ to $x_N=x_u$ and x_1 is obtained by direct interpolation. The numerical algorithm then consists of applying (7) and (8) for i= 2,..,N which will generate the adaptive grids. This method ensures that the grid spacing ratio is a very smooth function since this is the basic generator of the grids in this method. This, however, cannot be ensured if these grids were generated by simple interpolation methods. In that case, this sort of oscillation may make the numerical solution of partial differential equations on adaptive grids by the implicit method more difficult or even illposed.

3. RESULTS

We consider the function,

$$f(x) = \frac{1}{2}[1 + \tanh(10^3(x - 0.4))] \exp[-((x - 0.4)/0.2)^2], \tag{10}$$

on the interval [0,1]. As seen in Figure 1, this function is very steep in $0.4 < x < 0.8$. In order to resolve the sharp gradient there, we use the following form of $w(x;k)$ which is a very good candidate for resolving sharp gradients:

$$w(x;k) = (1.0 + k f_x{}^2)^{-1/2}. \tag{11}$$

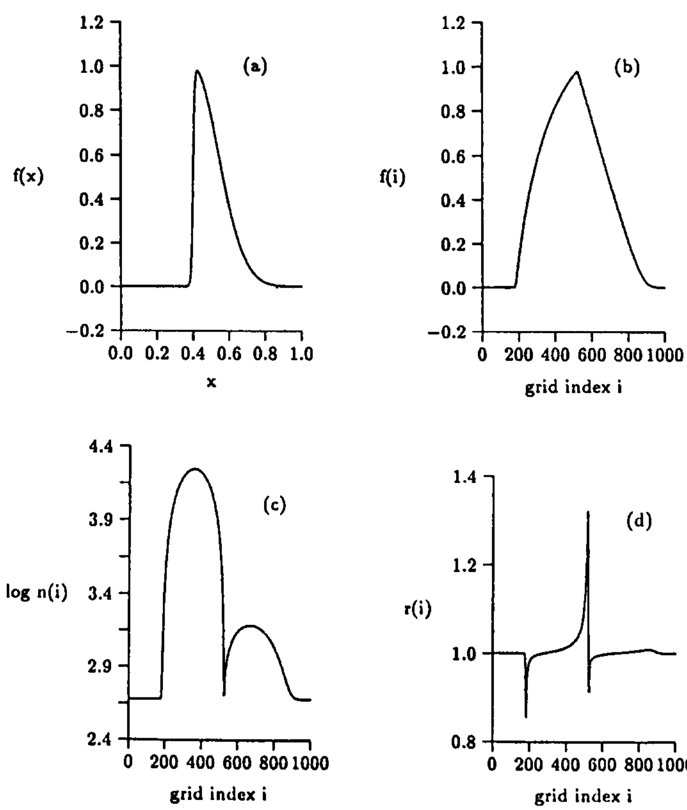

Figure 1. Distribution of 1000 adaptive grid points. (a) The test function (10) in physical space; (b) The test function in grid index space; (c) Resolution $\log(n)$ as a function of grid index; (d) grid spacing ratio r as a function of grid index i.

Figure 1 shows the function in x and ζ space. It also shows the grid spacing ratio and the resolution $n(i) = \dfrac{1}{x_i - x_{i-1}}$ as a function of the grid index (ζ). In our computation we have the used $N = 1000$ grid points and $k = 1$ in (11). As seen from our calculation, the sharp gradients are well resolved and we obtain a peak resolution 2×10^4, equivalent to having 2×10^4 uniformly spaced grid points.

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