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## New bounds for stabilizing Hele–Shaw flows

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### Abstract

We consider the problem of displacement processes in a three-layer fluid in a Hele–Shaw cell modeling enhanced processes of oil recovery by polymer flooding. The middle layer sandwiched between water and oil contains polymer-thickened water. We provide lower bounds on the length of the intermediate layer and on the amount of polymer in the middle layer for stabilizing the leading front to a specified level. We also provide an upper bound on the growth rate of instabilities for a given viscous profile of the middle layer.

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### 1. Introduction

In oil recovery by secondary displacement processes, a viscous oil in a porous medium is displaced by the injection of another less viscous immiscible fluid, usually water. The sharp interface (a contact discontinuity), within the Hele–Shaw model approximation, separating oil and water suffers from Saffman–Taylor instability [1] which is one of the sources of poor oil recovery as the moving unstable interface fails to sweep the oil efficiently before the interface breaks at the production well, thereby producing water instead of oil at breakthrough. The surface tension reduces the instability only to some extent which is not sufficient to improve oil recovery. In order to contain this instability to a meaningful level for improving oil recovery process before breakthrough, various tertiary displacement processes

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are employed. One of these processes involves displacing oil first with some polymer-thickened water (to be called ‘polysolution’ henceforth) followed by pure water. Such polymer-flooding processes have been addressed and studied in the 1980s by Gorell and Homsy [2] and Daripa et al. [3–5]. More recently, they have been studied by Carasso and Paşa [6–8].

The polymer-flooding processes basically involve three-layer fluid with an intermediate layer (to be called the ‘I.L.’ henceforth) of finite thickness containing polysolution. The viscosity of the polysolution depends on the concentration of polymer. A uniform distribution of polymer gives an I.L. of polysolution of uniform viscosity. Thus the polysolution is less viscous than oil but more viscous than water. On the other hand, a non-uniform distribution of polymer concentration gives an I.L. with variable viscosity. The effects of constant as well as variable viscosity I.L. in stabilizing porous media and Hele–Shaw flows have been studied in great detail numerically by Daripa et al. [3–5]. The effect of a variable viscosity I.L. in stabilizing Hele–Shaw flows has been studied by Gorell and Homsy [2] and Carasso and Paşa [6–8]. These studies concentrate on computing the growth rate of instabilities for a given length of the I.L. and a specified viscosity profile in the I.L.

In this work, we first provide an upper bound on the interfacial growth rate of instabilities in the presence of an I.L. We then make use of this upper bound to provide lower bounds on the length of the I.L. and the amount of polymer required for suppressing the Saffman–Taylor instabilities to a prescribed level. We also provide the corresponding viscosity of the polysolution at the leading front displacing the fluid as well as the bounds on the admissible viscosity profiles of the polysolution in the intermediate layer. In Section 2, we review the basics of the problem formulation from the literature but within a framework that helps in a concise derivation and presentation of our results in Sections 3 and 4. We discuss the relevance of our results and finally conclude in Section 5.

## 2. Preliminaries

### 2.1. The basic equations for the polymer model

The model we consider here within the Hele–Shaw approximation consists of three regions in the plane  $Ox_1y_1$ : a near-half-plane of water (region 1) stretching to  $x_1 \rightarrow -\infty$ , a similar one of oil (region 2) stretching to  $x_1 \rightarrow \infty$ , with a thin layer of polysolution in between (region 3). The polysolution in the intermediate layer I.L. (region 3) is assumed to have a variable viscosity  $\mu(x_1, y_1, t)$ , while  $\mu_1$  and  $\mu_2$  are the constant viscosities of the water and oil phases respectively. The fluid velocity  $(u, v)$  of the water phase upstream,  $x_1 \rightarrow -\infty$ , is considered to be  $(U, 0)$ , and the pressure is denoted by  $P$ . The relevant equations for this flow are then given by

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y_1} = 0, \quad x_1, y_1 \in \mathbb{R}, \quad (1)$$

$$\frac{\partial P}{\partial x_1} = -\mu u, \quad \frac{\partial P}{\partial y_1} = -\mu v, \quad x_1, y_1 \in \mathbb{R}, \quad (2)$$

$$\frac{D\mu}{Dt} = 0, \quad x_1 \in \text{I.L.}, y_1 \in \mathbb{R}, \quad (3)$$

with  $\mu$  in Eq. (2) defined as  $\mu = \mu_1$  in region 1 and  $\mu = \mu_2$  in region 2. The first equation above is the continuity equation, the second the Darcy law, and the third arises from the fact the polymer is merely advected by the aqueous phase and the polymer concentration is an invertible function of the viscosity.

A basic solution of the above system (1)–(3) in the  $(x_1, y_1)$  coordinates is given by

$$u = U, \quad v = 0, \quad \mu = \mu_0(x_1 - Ut), \tag{4}$$

$$P = -U \int_{x_0}^{x_1} \mu(s - Ut) ds = P_0(x_1, t). \tag{5}$$

In (4), the function  $\mu_0$  is an arbitrary basic viscosity profile which is a function of  $(x_1 - Ut)$  only. This function will be assumed to have all the properties stated below for  $\mu(x_1, y_1, t)$ . The above solution allows one to consider the intermediate region I.L. of length  $L$ , which at the initial moment is on the left of the origin. Thus, the region  $x_1 < Ut - L$  is filled with water, the I.L.  $Ut - L < x_1 < Ut$  with polysolution, and the region  $x_1 > Ut$  with oil. The three-layer fluid system is moving with the water velocity  $U$ . Two planar interfaces exist, one at  $x_1 = Ut - L$  (water–polysolution) and the other at  $x_1 = Ut$  (polysolution–oil). The variable viscosity  $\mu(x_1, y_1, t)$  in the I.L. layer at any time  $t \geq 0$  has the following properties: (i)  $\partial\mu/\partial x_1 \geq 0$  (i.e. viscosity increases monotonically from the left interface at  $x_1 = -L$  to the right interface at  $x_1 = 0$  in the intermediate region), (ii)  $\mu_2 > \min_{x_1} \mu(x_1, t) \geq \mu_1$ , and (iii)  $\mu_1 < \max_{x_1} \mu(x_1, t) \leq \mu_2$ . These properties of  $\mu$  in the I.L. are assumed to be part of the model.

In a moving frame  $x = x_1 - Ut, y = y_1$ , the water–polysolution part is contained in the region  $-L < x < 0$ , and the interfaces are at  $x = -L$ , and  $x = 0$ . In order to study linear stability of the basic solutions (4) and (5) in the moving frame, we consider an initial disturbance which slightly displaces the planar interfaces at  $x = -L, x = 0$  to non-planar configurations. We consider the perturbed solutions

$$u(x, y, t) = U + \epsilon\tilde{u}, \quad v(x, y, t) = \epsilon\tilde{v}, \quad P(x, y, t) = P_0(x) + \epsilon\tilde{p}, \quad \mu = \mu_0(x) + \epsilon\tilde{\mu}, \tag{6}$$

where  $\epsilon$  is a small parameter. We write Eqs. (1) through (3) in the above moving frame and then substitute (6) in these modified equations. We equate to zero the coefficients of the small parameter  $\epsilon$  to obtain the following linearized equations for  $\tilde{u}, \tilde{p}$ , and  $\tilde{\mu}$ :

$$\frac{\partial\tilde{u}}{\partial x} + \frac{\partial\tilde{v}}{\partial y} = 0, \quad x, y \in \mathbb{R}, \tag{7}$$

$$\frac{\partial\tilde{p}}{\partial x} = -\tilde{\mu}U - \mu_0\tilde{u}, \quad \frac{\partial\tilde{p}}{\partial y} = -\mu_0\tilde{v}, \quad x, y \in \mathbb{R}, \tag{8}$$

$$\frac{\partial\tilde{\mu}}{\partial t} + \tilde{u}\frac{d\mu_0}{dx} = 0, \quad -L < x < 0. \tag{9}$$

We study the temporal evolution of arbitrary perturbations by the method of normal modes. Hence, we consider a typical wave component of the form

$$(\tilde{u}, \tilde{p}, \tilde{\mu}) = (f(x), \psi(x), \phi(x))e^{(iky+\sigma t)}, \tag{10}$$

where  $k$  is a real axial wavenumber, and  $\sigma$  is the growth rate which could be complex. We substitute (10) in (7) through (9), and obtain equations involving  $f(x), \psi(x)$ , and  $\phi(x)$ . By appropriately cross-differentiating these equations for  $f(x), \psi(x), \phi(x)$ , we obtain the following equation for  $f(x)$  in the intermediate region:  $-L < x < 0$ ,

$$-(\mu_0 f_x)_x + k^2\mu_0 f = k^2 \frac{U}{\sigma} (\mu_0)_x f, \quad x \in (-L, 0), \tag{11}$$

which governs the linear stability of the basic solution (4) and (5). Above and below, we use the notation  $f_x = df/dx$ .

Two boundary conditions required to solve the above equation are obtained from the conditions that hold at the interfaces. At each of these interfaces, the following conditions must hold (see [9]).

- (a) Kinematic condition: Fluid particles on both sides of an interface must move with the interface without the two fluids occupying the same point at the same time and without a cavity forming between the fluids. This is equivalent to the statement that the velocity of the fluid particles normal to the interface is continuous across the interface and is equal to the material derivative of the interfacial disturbance.
- (b) Dynamic condition: The jump in normal stress of the fluid across an interface is balanced by the surface tension force at the interface.

From these conditions, after some manipulations (see [6] for their derivation) we obtain

$$\mu_0^+(-L)f_x(-L) = f(-L) \left\{ \mu_1 k + \frac{Uk^2}{\sigma} [\mu_1 - \mu_0^+(-L)] + \frac{Sk^4}{\sigma} \right\}, \quad (12)$$

$$\mu_0^-(0)f_x(0) = f(0) \left\{ -\mu_2 k + \frac{Uk^2}{\sigma} [\mu_2 - \mu_0^-(0)] - \frac{Tk^4}{\sigma} \right\}, \quad (13)$$

where  $T, S$  are the surface tension coefficients at  $x = 0$  and  $x = -L$  interfaces respectively, and superscripts ‘-’ and ‘+’ denote the “right” and “left” limit values respectively. The dispersion relation associated with solving (11) subject to (12) and (13) depends on the viscosity profile  $\mu_0(x)$ ,  $-L < x < 0$ , and hence allows calculation of the optimal viscosity profile, i.e., the viscosity  $\mu_0$  for which the growth rate  $\sigma$  is smallest over all modes.

Without the intermediate region (i.e., with zero polymer concentration),  $\mu_0(x)$  is a step function with jump at  $x = 0$  and  $L \rightarrow \infty$ . In this case, Eq. (11) becomes  $f_{xx} + k^2 f = 0$ ,  $-\infty \leq x < 0$ , which has solution  $f(x) = Ae^{kx}$ ,  $-\infty \leq x < 0$  that decays to zero in the far field. Using this solution with the relevant boundary condition (13), one obtains the well known dispersion relation for the two-layer Saffman–Taylor instability:

$$\sigma_s = \frac{(\mu_2 - \mu_1)Uk - Tk^3}{\mu_2 + \mu_1}, \quad (14)$$

where subscript ‘s’ stands for Saffman–Taylor instability. The maximum growth rate  $\sigma_{sm}$  is associated with wavenumber  $k_{sm}$  which is given by

$$\sigma_{sm} = \sigma(k_{sm}) = \frac{2(\mu_2 - \mu_1)U}{3(\mu_2 + \mu_1)} k_{sm}, \quad k_{sm} = \frac{1}{\sqrt{3}} \sqrt{\frac{(\mu_2 - \mu_1)U}{T}}. \quad (15)$$

We use  $\sigma_{sm}$  and  $k_{sm}$  to obtain the following non-dimensional quantities:

$$\begin{cases} \sigma^* = \frac{2\sigma}{3\sqrt{3}\sigma_{sm}}, & \mu^*(x) = \mu_0(x)/\mu_1, & k^* = k/(\sqrt{3}k_{sm}), & \mu_2^* = \mu_2/\mu_1, \\ x^* = \sqrt{3}k_{sm}x, & L^* = \sqrt{3}k_{sm}L, & f^*(x) = f(x)/U, & \lambda^* = 1/\sigma^*. \end{cases} \quad (16)$$

In the case of zero polymer concentration, the relations in (14) and (15) imply that the growth rate takes a maximum (non-dimensional) value  $\sigma_{sm}^*$  at a wavenumber  $k_{sm}^*$ .

$$\sigma_{sm}^* = \frac{2}{3\sqrt{3}} \approx 0.38, \quad k_{sm}^* = \frac{1}{\sqrt{3}}. \quad (17)$$

Below we consider the case of continuous viscosity across the water–polysolution interface at  $x^* = -L$ . We recall here that the polysolution in the I.L.,  $-L < x < 0$ , is an aqueous phase with polymer dispersed in it in such a way that the polymer concentration, and hence the viscosity, increases monotonically from  $x = -L$  to  $x = 0$ . Now, if we set up the model so that polymer concentration starts with zero concentration from the left interface, then essentially we have water on either side of the location  $x = -L$  which, in this special case, can be called a “pseudo-interface”. Therefore, there cannot be any surface tension at  $x = -L$ . Therefore, it is physically consistent here to consider that the surface tension ( $T$ ) acts only on the polysolution–oil interface at  $x^* = 0$  where we will have different liquid even if viscosities on either side were to be same. The linear stability of the polysolution–oil interface for the above set-up of the model is governed by the following Sturm–Liouville problem (omitting  $*$ ):

$$\begin{cases} -(\mu f_x)_x + k^2 \mu f = \lambda k^2 \beta \mu_x f, & x \in (-L, 0) \\ f_x(0) = (\lambda a + b) f(0), \\ f_x(-L) = k f(-L), \end{cases} \tag{18}$$

with

$$\begin{cases} a = k^2 \beta \{\mu_2 - \mu(0) - k^2(\mu_2 - 1)\} / \mu(0) \\ b = -k \mu_2 / \mu(0), \quad \beta = (\mu_2 + 1) / (\mu_2 - 1). \end{cases} \tag{19}$$

The two boundary conditions in (18) are scaled versions of dynamic boundary conditions (12) and (13) using  $S = 0$  and  $\mu_1 = \mu_0^+(-L)$ . Note that eigenvalue  $\lambda$ , which is the inverse of the growth rate (see (16)), appears in one of the boundary conditions which make this problem for a given  $\mu(x)$  non-trivial to solve for the dispersion relation  $\sigma(k)$  or for the upper bound of the growth rate. However, a numerical approach to obtaining the upper bound of the growth rate is feasible, which we address next. To this end, we note that total amount of polymer  $C$ , without any loss of generality, is defined by the following formula:

$$C = \int_{-L}^0 \mu(x) \, dx. \tag{20}$$

### 2.2. Estimate of the growth rate

A finite-difference approximation of the problem defined by (18) and (19) (see [6]) leads to

$$M \bar{f} = \lambda_{ap} \bar{f}, \tag{21}$$

where  $\bar{f} = (f_0, f_1, \dots, f_N)$ ,  $f_i \approx f(x_i)$ ,  $x_i = -ih$ ,  $h = L/N$ ,  $i = 0, 1, \dots, N$ , and  $\lambda_{ap}$ , for which the discrete system (21) has exact solutions, is the numerical approximation of  $\lambda$ . The corresponding approximate value of the growth rate is denoted by  $\sigma_{ap} = 1/\lambda_{ap}$ . Gerschgorin’s localization theorem gives localization of eigenvalues  $\lambda_i$  of the matrix  $M$  from which one obtains the inequalities (see [6] for details)

$$-\frac{b}{a} \leq \lambda_{ap} = 1/\sigma_{ap} \leq \frac{2 - bh}{ah}, \quad \frac{\mu}{\beta \mu'} < \lambda_{ap} = 1/\sigma_{ap}. \tag{22}$$

Since

$$-\frac{a}{b} = H(k, \mu(0)) = \frac{k\beta}{\mu_2} \{\mu_2 - \mu(0) - k^2(\mu_2 - 1)\}, \tag{23}$$

the left inequality of (22)<sub>1</sub> and inequality (22)<sub>2</sub> yield

$$\sigma_{ap} \leq H(k, \mu(0)), \quad \sigma_{ap} < \frac{\beta \mu'_i}{\mu_i}, \quad i = 1, 2, \dots, (N - 1), \quad (24)$$

which leads to the following upper bound  $\sigma_m$  for the growth rate  $\sigma_{ap}$  such that  $0 < \sigma_{ap} \leq \sigma_m$ :

$$\sigma_m = \text{Max}_{\{k,i\}} \left\{ H(k, \mu(0)), \frac{\beta \mu'_i}{\mu_i} \right\}, \quad i = 1, 2, \dots, (N - 1), \quad (25)$$

where a prime denotes a partial derivative with respect to  $x$ , and  $\mu_i \approx \mu(x_i)$ . At  $k = k_m$ ,  $H(k, \mu(0))$  attains its maximum value  $F(\mu(0)) = \text{Max}_{\{k\}} \{H(k, \mu(0))\}$  given by

$$F(\mu(0)) = \frac{2\beta}{3\mu_2} \frac{(\mu_2 - \mu(0))^{3/2}}{\sqrt{3(\mu_2 - 1)}} > 0, \quad k_m = \sqrt{\frac{\mu_2 - \mu(0)}{3(\mu_2 - 1)}}. \quad (26)$$

Hence Eq. (25) becomes

$$\sigma_m = \text{Max}_{1 \leq i \leq N-1} \left\{ F(\mu(0)), \frac{\beta \mu'_i}{\mu_i} \right\}. \quad (27)$$

Note from (23) and (24) that the unstable modes ( $\sigma_{ap}(k) > 0$ ) have wavenumbers  $k \in (0, k_{cr})$  with  $k_{cr} < k_c$  where  $k_c = \sqrt{[\mu_2 - \mu(0)]/[\mu_2 - 1]}$ , which in dimensional form becomes  $k_c = \sqrt{\frac{U}{T}(\mu_2 - \mu(0))}$ . In the limit of  $\mu(0) \rightarrow \mu_1$  (two-layer case),  $k_c$  is same as the critical wavenumber for the two-layer case (see (15)) and therefore  $k_{cr} = k_c$  in this limit which supports our estimate  $k_{cr} < k_c$  for the three-layer case.

The convergence of the solutions of (21) to the exact solutions of (18) and (19) has been proved in [7] and therefore the above inequalities and the estimates (25) also hold for the exact value of the growth rate.

Below, we consider only those viscosity profiles in the intermediate region for which the following relation holds:

$$(\beta \mu')/\mu \leq F(\mu(0)). \quad (28)$$

Then the following estimate of  $\sigma_m$  follows from (27) and (28):

$$\sigma_m = F(\mu(0)). \quad (29)$$

An improvement in stability requires that the upper bound,  $\sigma_m$ , of the growth rate for this three-layer fluid case be less than that ( $\sigma_{sm}$ ) for the two-layer (Saffman–Taylor) case. This then implies that

$$\sigma_m < \sigma_{sm} \Rightarrow F(\mu(0)) < \frac{2}{3\sqrt{3}}, \quad (30)$$

which, together with (26), in turn implies

$$\frac{\beta(\mu_2 - \mu(0))^{3/2}}{\mu_2 \sqrt{(\mu_2 - 1)}} < 1. \quad (31)$$

**Remark 1.** When  $\mu(0) = \mu_2$ ,  $\sigma_m \rightarrow 0$  and hence  $\mu' = 0$  (see (28)), implying that the viscosity is constant at  $\mu_2$  in the middle layer which violates the continuity requirement for the viscosity at the interface  $x = -L$ . Thus the value of  $\mu(0) = \mu_2$  is not admissible any more for viscosity profiles subject

to the constraint (28). Henceforth, we consider values of  $\mu(0) \in [1, \mu_2)$  excluding the possibility of  $\mu(0) = \mu_2$ .

**Remark 2.** In this remark, all variables appear in dimensional form. If the middle layer has a constant viscosity  $\mu = \mu_c$  such that  $\mu_1 < \mu_c < \mu_2$ , then the upper bound  $\sigma_{cm}$  on the growth rate of the leading interface will be given by the Saffman–Taylor-type formulae within a very good approximation, at least for large  $L$ . Therefore, we can write (replace  $\mu_1$  in (15)<sub>1</sub> with  $\mu_c$ )

$$\sigma_{cm} = \frac{2}{3\sqrt{3}} \frac{(\mu_2 - \mu_c)^{3/2} U^{3/2}}{\mu_2 + \mu_c T}. \tag{32}$$

This should be compared with the upper bound  $\sigma_m$  given above for the variable viscosity profile constrained by (28) (see also the next section). Below, we use the new notation  $\sigma_{nm}$  for the dimensional form of the upper bound  $\sigma_m$ . It then follows from using (26)<sub>1</sub> in (29) and then going back to the dimensional form using (16) that

$$\sigma_{nm} = \frac{2}{3\sqrt{3}} \frac{(\mu_2 - \mu(0))^{3/2} U^{3/2}}{\mu_2 T}. \tag{33}$$

Therefore, a measure of control of the instability of the leading interface by using variable viscosity rather than constant viscosity is given by the ratio of above two bounds, which simplifies, after using the formula (32) and (33) for these bounds, to

$$\frac{\sigma_{nm}}{\sigma_{cm}} = \left( \frac{\mu_2 + \mu_c}{\mu_2} \right) \left( \frac{\mu_2 - \mu(0)}{\mu_2 - \mu_c} \right)^{3/2}. \tag{34}$$

For the case when  $\mu_c = \mu(0)$  which corresponds to the use of significantly more polymer in the constant viscosity case than in the variable viscosity case, it is expected on physical grounds that the leading interface will be much less unstable for the constant viscosity case than for the variable viscosity case, i.e.,  $\sigma_{cm} < \sigma_{nm}$ , which is borne out by the above formula (34), because when  $\mu_c = \mu(0)$ , the above formula gives

$$\frac{\sigma_{nm}}{\sigma_{cm}} = \left( 1 + \frac{\mu(0)}{\mu_2} \right) > 1. \tag{35}$$

For the case when  $\mu_c = \mu_1$ , which is more akin to the case for polymer flooding because it corresponds to the use of no polymer in the constant viscosity case and a finite amount of polymer in the variable viscosity case, it is then expected on physical grounds that the leading interface will be more unstable for the constant viscosity case than for the variable viscosity case, i.e.,  $\sigma_{cm} > \sigma_{nm}$ , which is also borne out by the above formula (35), because when  $\mu_c = \mu_1$ , the above formula (35) gives

$$\frac{\sigma_{nm}}{\sigma_{cm}} = \left( 1 + \frac{\mu_1}{\mu_2} \right) \left( \frac{\mu_2 - \mu(0)}{\mu_2 - \mu_1} \right)^{3/2}, \tag{36}$$

which is usually a small number, much less than one, in most circumstances with water viscosity ( $\mu_1$ ) much less than the oil viscosity ( $\mu_2$ ) and the viscosity  $\mu(0)$  much closer to the oil viscosity. For example, with  $\mu_2/\mu_1 = 10$  and  $\mu(0)/\mu_1 = 9$  which are realistic in many cases (see [4]), the above formula (36) gives  $\sigma_{nm}/\sigma_{cm} \approx 0.04$  which amounts to a 25-fold reduction in the growth rate of the leading interface! Therefore, it is possible to contain the leading interface instability significantly by polymer flooding the intermediate layer in an appropriate manner.

### 3. Improved restrictions for $\mu(0)$

The estimate (25) holds only for the “interior” values of the viscosity in the intermediate region. Therefore we consider the restriction (28) in the following form:

$$\mu'(z)/\mu(z) \leq F(\mu(0))/\beta, \quad -L < z < 0. \quad (37)$$

By integration of the above relation for  $z \in (-L, x)$ , we have

$$\mu(x) \leq \exp \left\{ (x + L) \frac{F(\mu(0))}{\beta} \right\}, \quad x \in (-L, 0). \quad (38)$$

This relation is obtained using the continuity of the viscosity at  $x = -L$ , i.e.,  $\mu(-L) = 1$ . Using the above inequality in (20) and integrating, we obtain the following lower bound for the “optimal” injection length  $L$  in terms of  $C$  and  $\mu(0)$ :

$$L \geq \frac{\beta}{F(\mu(0))} \ln \left\{ \frac{CF(\mu(0))}{\beta} + 1 \right\}. \quad (39)$$

The inequality (38) also holds at  $x = 0$  in the intermediate layer (see [6,7]) which then leads to

$$\mu(0) \leq \exp \left\{ L \frac{F(\mu(0))}{\beta} \right\}. \quad (40)$$

From this we obtain

$$L \geq \frac{\beta}{F(\mu(0))} \ln\{\mu(0)\}, \quad (41)$$

which is a “second” lower bound on the optimal injection length in terms of  $\mu(0)$  only, unlike the first bound given in (39). However, the inequality (39), in conjunction with (41), is useful for obtaining a lower bound on the amount  $C$  of polymer which we show below.

The relations (39) and (41) are used to obtain lower bounds on  $C$  and  $L$  in terms of  $\mu(0)$ . If the amount of polymer  $C$  satisfies

$$C \geq \beta\{\mu(0) - 1\}/F(\mu(0)), \quad (42)$$

then

$$\frac{\beta}{F(\mu(0))} \ln \left\{ \frac{CF(\mu(0))}{\beta} + 1 \right\} \geq \frac{\beta}{F(\mu(0))} \ln\{\mu(0)\}, \quad (43)$$

which in conjunction with (39) and (41) gives

$$L \geq \frac{\beta}{F(\mu(0))} \ln \left\{ \frac{CF(\mu(0))}{\beta} + 1 \right\} \geq \frac{\beta}{F(\mu(0))} \ln\{\mu(0)\}. \quad (44)$$

Therefore the lower bound (39) of  $L$  is compatible with (41) if  $C$  satisfies (42). The relations (30), (42) and (44) give direct estimates for  $\sigma$ ,  $C$  and  $L$  in terms of  $\mu(0)$ . Thus, for a desired level of improvement in the stability (i.e. for a given bound on  $\sigma_m$ ), the amount of polymer  $C$  and the length  $L$  of the intermediate region have lower bounds which are obtained below.



#### 4. Lower bounds on $L$ and $C$

For stabilization of the flow, we prescribe that the upper bound,  $\sigma_m$ , of the growth rate be a fraction of  $\sigma_{sm}$  given by

$$\sigma_m = p\sigma_{sm}, \quad 0 < p \leq 1, \tag{45}$$

so that  $(1 - p) \times 100$  can be viewed as a percentage gain in improvement of stability over the Saffman–Taylor case. The above relation then gives

$$F(\mu(0)) = \frac{2p}{3\sqrt{3}} \implies \frac{2\beta}{3\mu_2\sqrt{3}} \frac{(\mu_2 - \mu(0))^{3/2}}{(\mu_2 - 1)^{1/2}} = \frac{2p}{3\sqrt{3}}. \tag{46}$$

From this we obtain

$$\mu(0) = \mu_2 - (p^2\mu_2^2(\mu_2 - 1)/\beta^2)^{1/3}. \tag{47}$$

Note that  $1 \leq \mu(0) < \mu_2$ . The right inequality is transparent. To see the left inequality, first note that  $(\mu_2 + 1)^2 \geq p^2\mu_2^2$  which implies  $(\mu_2 - 1)^2 \geq (p^2\mu_2^2/\beta^2)$ . If we multiply both sides of this inequality by  $(\mu_2 - 1)$  and then take its cubic root, we get  $(\mu_2 - 1) \geq (p^2\mu_2^2(\mu_2 - 1)/\beta^2)^{1/3}$ . Using this in (47) shows that  $\mu(0) \geq 1$ . Formula (47) gives  $\mu(0)$  corresponding to a given improvement in stability over the Saffman–Taylor case.

Finally, we use (47) in (42) and (44) to obtain the minimum values of  $C, L$  in terms of  $\mu_2, p$ :

$$C_{\min} = \frac{3\sqrt{3}\beta}{2p} \left\{ (\mu_2 - 1) - \left( \frac{p^2\mu_2^2}{\beta^2} (\mu_2 - 1) \right)^{1/3} \right\}, \tag{48}$$

$$L_{\min} = \frac{3\sqrt{3}\beta}{2p} \ln \left\{ \mu_2 - \left( \frac{p^2\mu_2^2}{\beta^2} (\mu_2 - 1) \right)^{1/3} \right\}. \tag{49}$$

The formulae (47), (48) and (49) give  $\mu(0), C$  and  $L$  corresponding to the improvement (45) of the stability in terms of the viscosity ratio  $\mu_2$  (recall that this is normalized by  $\mu_1$ ). For a desired level ( $p$  in (45)) of improvement of stability over the Saffman–Taylor case, one should choose the polymer amount  $C > C_{\min}$ , the length  $L > L_{\min}$ , and the viscosity profiles (38) consistent with  $\mu(0)$  given by (47).

#### 5. Discussion and conclusions

In this work, we have obtained some new bounds on several quantities based on linearized stability analysis and assuming that the viscosity is continuous at the trailing interface. The linearized stability analysis used in this work can be applied in principle, though not without technical complications and cumbersome algebra, to the general case where the trailing interface can also have a jump in viscosity. However, this falls outside the scope of the study undertaken here and may be addressed in the future.

The results obtained here based on linear theory are not intended for applications in the nonlinear regime, though it will be very interesting if these results hold in the nonlinear regime. It is a well known fact that the late-time evolution of fronts either in a Hele–Shaw cell or in porous media is a highly nonlinear phenomenon and no theory based on linear analysis is capable of describing the ultimate frontal dynamics for a given set of parameters except for a short duration during early stages of the evolution.

In some cases, if it does, it is more an exception than the rule. On the other hand, it is very important to realize that the ultimate frontal dynamics of viscosity driven instability is described by, among other factors, the viscosity contrast across the front. These are well established facts from physical experiments as well as numerical simulations. The severity of the fully fledged fingering phenomenon associated with the ultimate frontal dynamics in the Hele–Shaw cell is directly influenced by the viscosity ratio at the front: the higher the viscosity ratio (ratio of viscosity of the displaced fluid to the viscosity of the displacing fluid), the more severe the fingering phenomenon. Fortunately, this fact of ultimate frontal dynamics in the nonlinear regime has a parallel in the exact linear theory in a qualitative sense: namely, the higher the viscosity ratio at the front, the more severe the growth rates of individual modes. This analogy in no way implies that the linear stability dictates ultimate growth of fronts. This qualitative analogy between linear (short-time) and nonlinear (long-time) dynamics and the fact, validated by numerical and physical experiments (see [4]), that the instability controlling strategy derived from linear theory translates into the nonlinear regime are at the very heart of the merits of the results based on linear theory, because such results can be obtained either exactly or easily with little or no error through using well-established mathematical tools of linear theory.

It is important to realize that no assumption of the sort that the linear stability dictates ultimate growth of fronts is implied by our derivations of the various bounds here. Rather, results obtained based on linear theory can be used effectively in parameter space (viscosity ratio) to mitigate early-time growth of individual modes whose evolution through subsequent nonlinear interaction of these controlled instabilities can mitigate the extent of the fingering phenomenon at late times. Such results can also be used in decision making processes (see below) and also for setting up physical experiments to assess the connection between reality and predictions based on linear theory for the nonlinear regime. The results of the kind provided here are quantitative and hence very precise. This allows one to estimate quantities required for starting relevant projects or for setting up scientific experiments. If one were to engage in enhanced oil recovery by polymer flooding of an oil-field, several questions would immediately arise in the context of what is discussed in this work. Some are what the optimal amount of polymer to use is, what the viscosity in the polysolution at the interface directly in contact with the oil should be, how long the polymer flooding should be continued before reverting back to pure water injection, and so on. Similar questions would arise if one were even to set up a model experiment for such purposes to experimentally assess the actual merits of the whole process, since ultimately the whole process is highly nonlinear. Such questions are easily answered with the help of quantitative results given here and hence help in the decision making process. In the light of these facts, we address the utility of our results below.

We have given an upper bound on the growth rate (29), and lower bounds on the injection length  $L$  (see (44)) and on the total amount of polymer  $C$  (see (42)): all in terms of the viscosity ratio  $\mu(0)$  (non-dimensionalized by the viscosity  $\mu_1$ ) for the optimal viscous profile given by (39). So, once the choice of  $\mu(0)$  has been made, one can use these estimates to calculate the other quantities ( $L$ ,  $C$ ) which are necessary to set up an experiment, physical or numerical. Moreover, using our estimate (47), one can even calculate the improvement  $p$  in stability over the two-layer Saffman–Taylor case which should allow one, in principle, to predict the average location of the front, at least at early stages of evolution, and then compare this with actual experimental or numerical calculations. Such comparisons are at times the catalyst for generating new ideas for improving estimates or modifying estimates based on some new data (perhaps empirical at times) to extend their validity even for late-time evolution of fronts. Without such estimates, setting up experiments to achieve similar goals is a much more expensive proposition in

most cases, as these have to be based on trial and error through many numerical or physical simulations, and sometimes such tasks cannot even be performed repeatedly, e.g. in a real oil-field.

In the above, the choice of  $\mu(0)$  enables the rest of the calculations required for the set-up. Alternatively, one can choose the desired level of improvement  $p$  in stability over the Saffman–Taylor case and then find  $\mu(0)$  using the estimate (47). The rest of the calculation then follows as above. In fact, one can then use estimates (48) and (49) to estimate the minimum values of  $C$  and  $L$  required for the set-up of the experiment.

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