Local and Nonlocal Solitary Waves of Boussinesq Equations

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Abstract

Local and non-local solitary waves of Boussinesq equations are considered. According to the classical weakly nonlinear theory of water waves, an illposed Boussinesq equation governs the propagation of long waves to leading order approximation and admits localized solitary wave solutions. The severe short wave instability of this equation casts real utility of this model. Necessity of smart filtering techniques is exemplified. Using dispersive regularization, it is shown numerically and theoretically that the regularized equation admits non-local solitary wave solutions with oscillating tails at infinity. Computational aspects of solving these equations and their implications are also considered.

1 Introduction

The Boussinesq equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}, \tag{1}$$

describes propagation of small amplitude long waves in both the positive and the negative x-directions in the weakly nonlinear long wave approximation of full water wave equations [7]. This equation arises in many physical systems including shallow water under gravity [10] and one dimensional nonlinear lattices [11]. This equation admits solitary wave solutions

$$u^s(x,t) = A\ \mathrm{sech}^2 \left\{ \sqrt{A/6}(x - ct) \right\}, \tag{2}$$

where A is the amplitude of the solitary wave, and $c = \mp\sqrt{1 + 2A/3}$ is the speed of the solitary wave. Such locally confined waves of permanent form indeed have been shown to exist by both rigorous analysis [1] and numerical simulations [2, 8] of full nonlinear water wave equations. These features of equation (1) are quite reminiscent of the properties of the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{3}$$

except that the KdV equation allows only one-directional wave propagation.

The Boussinesq equation (1) has pathological behavior in the sense that its linearized version about a constant state u_c has decaying as well as growing normal mode solutions periodic in x with wavelength λ, of the form $e^{\sigma t + ikx}$ where $k = \frac{2\pi}{\lambda}$. The linearized dispersion relation for this equation is given by

$$\sigma_{\mp} = \mp k\sqrt{k^2 - p'(u_c)}, \tag{4}$$

where $p(u) = u + u^2$. The equilibrium states in the elliptic region (i.e. $p'(u_c) = 1 + 2u_c < 0$) are unstable to all modes and the states in the hyperbolic region are unstable to modes $|k| > \sqrt{p'(u_c)}$. We are interested in the hyperbolic regime only. It is worthwhile to point out that numerical as well as theoretical issues in the supercritical, critical and subcritical regimes could be very different.

According to the dispersion relation (4), short-wave instability is given by

$$\sigma \sim k^2 \quad \text{as} \quad k \to \infty. \tag{5}$$

This catastrophic instability of equation (1) is non-physical and is a consequence of breakdown of the model at short wavelength.

This equation is usually regularized as follows

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxtt}. \tag{6}$$

Our modification of choice in this paper is different from this standard regularization. We regularize the illposed equation via singular perturbation as follows.

$$u_{tt} = (p(u))_{xx} + u_{xxxx} + \delta u_{xxxxxx}, \quad \delta > 0. \tag{7}$$

The sixth order derivative term in equation (7) provides dispersive regularization and is not an ad-hoc regularization term (see [5] for more details). This regularized equation (7) has decaying as well as growing normal modes, $e^{\sigma t+ikx}$, with the linearized dispersion relation about the constant state, u_c, given by

$$\sigma_{\mp} = \mp k\sqrt{k^2 - p'(u_c) - \delta k^4}, \quad \delta > 0. \tag{8}$$

It clearly shows the absence of short wave instability for fixed $\delta > 0$ and in fact, all modes

$$k > (1 + \sqrt{1 - 4p'\delta})/2\delta; \quad 4p'(u_c)\delta < 1 \tag{9}$$

are neutrally stable.

This short paper summarizes some key findings on these Boussinesq equations and also points out some links and analogies of the present work to modeling more complex systems. The presentation of numerical results has been kept to a bare minimum due to page limitation. Somewhat more elaborate and thorough numerical study of this problem with many numerical results can be found in Daripa and Hua [3] and in Daripa [4]. Theroretical results can be found in [5] and [6].

2 Numerical Method

Equations (1) and (7) are solved numerically in a finite domain, $a \le x \le b$, for $t > 0$ using a finite difference method with uniform grid spacings h in x and τ in t for various choices of δ and subject to initial data $u^s(x,0), u^s_t(x,0)$ and boundary conditions $u^s(a,t), u^s(b,t)$ where $u^s(x,t)$ is given by (2). The amplitude of the solitary wave is taken to be $A = 0.5$. The boundaries are carefully chosen so that they are far away from the support of the solitary wave for the duration of the computation. The details of this numerical method are described in Daripa and Hua [3] and in Daripa [4].

3 Numerical Results

Many numerical results on this problem can be found in [3]. Here we briefly present some key results. For $\delta = 0$ (illposed Boussinesq equation), instability is so violent that fine scale motions which initially appear are rapidly amplified making the calculations completely meaningless within a very short time. Use of larger mesh sizes inhibits active participation of the very high wavenumber modes but the effect of large truncation error in this case quickly deteriorates the accuracy of the numerical solutions. High precision calculations allow accurate computations for little longer but discretization error in the high wavenumber modes gets amplified and destroys the calculations. For better results, we needed to use smart filtering techniques (see Daripa [4] for details) in Fourier space to smooth the fine scale oscillations.

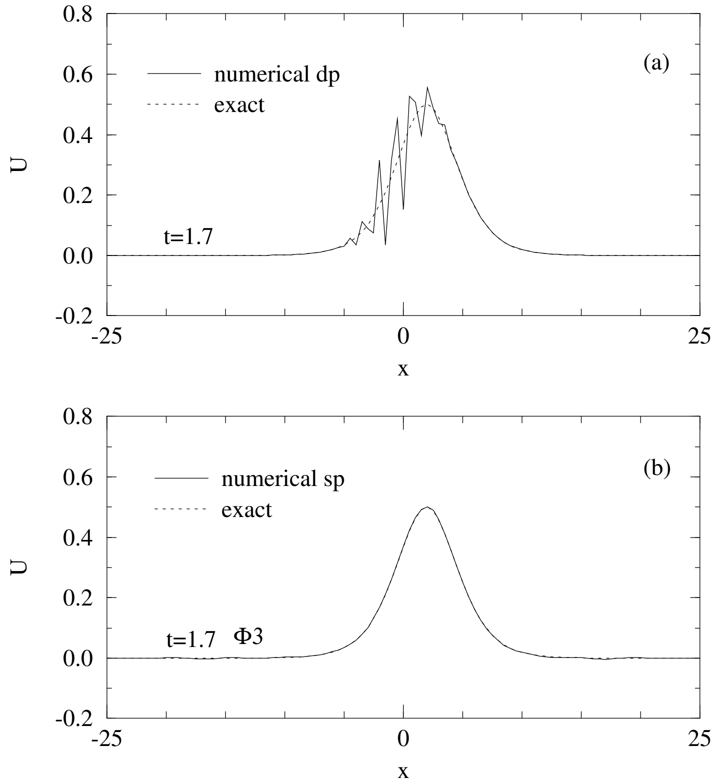

Figure 3.1: Comparison of numerical and exact solutions of equation 1 at the time level $t = 1.7$. The calculations were done in (a) **double precision** (15 digit arithmetics); (b) **single precision** (7 digit arithmetics) **with filter $\boldsymbol{\Phi_3}$ at filter level 5**. All the computations use $h = 0.5$, $\tau = 0.01$.

Figure 3.1(a) shows the numerical solution at $t = 1.7$ obtained in double precision calculations with no filter in use. A comparison of this numerical solution with the exact solution in this figure clearly shows spurious small scale ripples which are rather severe. Figure 3.1(b) shows the numerical solution at the same time level obtained using smart filtering in single precision calculation. The filter that has been used for this purpose is twice continuously differentiable, denoted by Φ_3, and construction and usefulness of this filter have been discussed in detail in [4]. Continuation of the numerical solution shown in figure 3.1(b) for longer time breaks down soon after $t = 4$ due to development of small oscillations.

Numerical experiments with equation (7) for various choices of δ have been computed and compared with the exact solitary wave solution (2) of the illposed Boussinesq equation ($\delta = 0$ case) in order to test the effectiveness of the regularization technique as well as to investigate the structural stability of the solitary wave (2) under dispersive perturbation of the illposed Boussinesq equation. Numerical results for various choices of $\delta \leq 1.0$ can be found in Daripa and Hua [3]. Numerical results are not shown here due to page limitation.

Two observations were made. First, numerical solutions for positive values of δ are good approximations to the exact solutions of the illposed equation (1) in the core of the wave. Secondly, oscillations of small amplitude appear at the tail ends of the solitary wave and this amplitude depends on the value of δ. Extensive numerical experiments led us to believe that oscillations far away from the core are not results of numerical artifacts but are rather due to singular perturbation of the illposed Boussinesq equation (1).

4 Theoretical Results

Motivated by the numerical results, we proved theoretically that the solitary waves (2) of equation (1) are structurally unstable under the singular perturbation of the dispersive type used in (7). In particular, we show in [5] the existence of non-local solitary waves for the regularized equation. An estimate of the size of these amplitudes which depends on the parameter δ can be found in [5]. Issue of singularity formation subject to small data with small Fourier support for the illposed Boussinesq equation has also been addressed in [6].

5 Discussion and Concluding Remarks

Here we have addressed the issue of regularization of illposed Boussinesq equation by singular (dispersive) perturbation as well as by smart filtering in Fourier space. We have also provided numerical and theoretical evidence of the existence of non-local solitary wave solutions (with dispersive (capillary) ripples at infinity) of the regularized Boussinesq equation. Issue of singularity formation in the illposed Boussinesq equation is also addressed.

It is also worthwhile to point out some links and analogies of the filtering and regularization techniques to subgrid scale turbulence modeling [9]. In subgrid scale modeling, equations of fluid dynamics (Navier-Stokes equations) are rewritten in spectral space making explicit distinction between explicit and subgrid scales. Evolution equations of these two scales are separated with spectral/eddy viscosity appearing only in the equation for subgrid scales. Similarly, in filtering and regularization techniques the modes are separated into two parts in spectral space: subcritical ($k < k_c$) and supercritical ($k > k_c$) modes, where the critical mode k_c is carefully selected based on machine precision and grid size (see [3],[4]).

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References

[1] C. J. Amick and J. F. Toland, (1981) *On solitary water waves of finite amplitude*, Arch. Rat. Mech. Anal. **76**, 9-95.

[2] J. G. B. Byatt-Smith and M. S. Longuet-Higgins, (1976) *On the speed and profile of steep solitary waves*, Proc. R. Soc. Lond. A **350**, 175-189.

[3] P. Daripa and W. Hua, (1999) *A Numerical Method For Solving an Ill-posed Boussinesq Equation Arising in Water Waves and Nonlinear Lattices*, Applied Math. Comp., 101/2-3, 159-207.

[4] P. Daripa, (1998) *Some Useful Filtering Techniques For Illposed Problems*, Jour. Comp. Appl. Math., 100, 161-171.

[5] P. Daripa and R. Dash, "Singular Perturbation of the Illposed Boussinesq Equation", To Be Submitted.

[6] P. Daripa and R. Dash, "On Singularity Formation in the Illposed Boussinesq Equation", To Be Submitted.

[7] R. S. Johnson, (1997) *A Modern Introduction to Mathematical Theory of Water Waves*, Cambridge Univ. Press.

[8] M. S. Longuet-Higgins and J. D. Fenton, (1974) *On the mass, momentum, energy and circulation of a solitary wave. II*, Proc. R. Soc. Lond. A **340**, 471-493.

[9] W. D. McComb, (1994) *The Physics of Fluid Turbulence*, Oxford Science Publications.

[10] G. B. Whitham, (1974) *Linear and Nonlinear Waves*, John Wiley & Sons, Inc.

[11] V. E. Zhakarov (1974) *On stochastization of one-dimensional chains of nonlinear oscillation*, Sov. Phys. JETP, 38(1), pp. 108-110.