Studies of Capillary Ripples in a Sixth-Order Boussinesq Equation Arising in Water Waves

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Abstract

In this paper, we study the sixth-order Boussinesq equation recently introduced by Daripa and Hua [Appl. Math. Comput. **101** (1999), 159–207]. This equation describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for Bond number less than but very close to 1/3. On the basis of far-field analyses and heuristic arguments, we show that the traveling wave solutions of this equation are weakly non-local solitary waves characterized by small amplitude fast oscillations in the far-field. We construct these solutions and provide estimates of the amplitude of the associated oscillations both analytically and numerically.

1 Introduction

In this paper, we study the singularly perturbed (sixth-order) Boussinesq equation

$$\eta_{tt} = \eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx} + \epsilon^2 \eta_{xxxxxx}, \tag{1.1}$$

where ϵ is a small parameter. This equation was originally introduced by Daripa & Hua [3] as a regularization of the ill-posed classical (fourth-order) Boussinesq equation which corresponds to $\epsilon = 0$ in equation (1.1). It is well known that the fourth-order Boussinesq equation possesses the traveling solitary wave solutions.

The physical relevance of equation (1.1) in the context of water waves was recently addressed by Daripa & Dash [4]. It was shown that this equation actually describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for Bond number less than but very close to 1/3. So, it is closely related to the fifth-order KdV equation originally derived by Hunter & Scherule [7] and subsequently studied by Akylas & Yang [1], Boyd [2], Grimshaw & Joshi [6], Pomeau et al. [8], and many others. The fifth-order KdV equation is restricted only to uni-directional propagating waves.

In this paper, we construct weakly non-local solitary wave solutions of the sixth-order Boussinesq equation (1.1) in the form of traveling waves by using analytical and numerical methods originally devised to obtain this type of solutions of the fifth-order KdV equation. We also obtain estimates of the amplitude of the oscillatory tails associated with these weakly non-local solitary waves.

2 Preliminaries

Since equation (1.1) has solitary wave solutions for $\epsilon = 0$, the natural question arises as to whether equation (1.1) also admits solitary wave solutions for small positive values of ϵ. Therefore, we seek traveling wave solutions of equation (1.1) in the form $\eta(x,t) = \eta(x - ct)$ where c is the phase speed (velocity) of the wave. Substituting it in equation (1.1) and using x for the new variable $x - ct$ yields

$$(1 - c^2)\eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx} + \epsilon^2 \eta_{xxxxxx} = 0. \tag{2.1}$$

The question now becomes whether equation (2.1) admits solutions which decay exponentially to zero as $x \to \pm\infty$ for any small positive value of ϵ. Since we are interested in bounded solutions of equation (2.1) as $x \to \pm\infty$, on integrating equation (2.1) twice and taking the constants of integration as zero, we obtain

$$(1 - c^2)\eta + \eta^2 + \eta_{xx} + \epsilon^2 \eta_{xxxx} = 0. \tag{2.2}$$

It can be easily shown that (see Dash & Daripa [5]) an approximate solution of equation (2.2) can be obtained as a regular asymptotic expansion in ϵ^2 in the form

$$\eta = \eta_0 + \epsilon^2 \big(-10\gamma^2 \eta_0 + \frac{5}{2}\eta_0^2 \big) + \cdots, \tag{2.3}$$

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where $\eta_0 = 6\gamma^2 \text{sech}^2(\gamma x)$ is the solitary wave solution of the fourth-order Boussinesq equation and γ is a free parameter characterizing the width of the wave. The phase speed c is related to γ by $c^2 - 1 = 4\gamma^2 + 16\epsilon^2\gamma^4 + O(\epsilon^4)$. The form of solution (2.3) implies that η is symmetric about $x = 0$ and decays down to zero exponentially as $x \to \pm\infty$. So, by the method of regular asymptotic analysis, we only get exponentially decaying solution in the far-field. However, as we will see below, the far-field analysis contradicts this.

If we assume that η is small in the far-field $x \to \pm\infty$, then equation (2.2) linearizes to

$$(1 - c^2)\eta + \eta_{xx} + \epsilon^2\ \eta_{xxxx} = 0 \quad \text{as} \quad x \to \pm\infty. \tag{2.4}$$

Equation (2.4) has solutions of the form $\eta = \exp(ipx)$ provided $\epsilon^2 p^4 - p^2 = (c^2 - 1)$. Since $|c| > 1$, this characteristics equation has two real roots (which correspond to the oscillatory behavior of η at infinity) and two purely imaginary roots (which correspond to decaying and growing behavior of η at infinity). For a local solitary wave, only the root which corresponds to the decaying behavior of η at infinity is acceptable. This then implies the necessity of three independent boundary conditions on η as $x \to \infty$, with three more as $x \to -\infty$, leading altogether to the necessity of six independent boundary conditions on η for a fourth-order differential equation (2.2). Therefore, we can not force η to vanish at both $x \to \infty$ and $x \to -\infty$. There will be an oscillatory behavior at least on one side at infinity with the general form given by

$$\eta = A_{\pm} \sin\left[\frac{q}{\epsilon}(x + \phi_{\pm})\right] \quad \text{as} \quad x \to \pm\infty, \tag{2.5}$$

where $q^2 = 1 + 4\epsilon^2\gamma^2 + O(\epsilon^4)$. Here $A_{\pm}$ and $\phi_{\pm}$ are the amplitude and phase shift constant of the oscillatory tails as $x \to \pm\infty$. For symmetric solutions, $A_+ = A_- = A$ and $\phi_+ = \phi_- = \phi$. Since $\frac{q}{\epsilon} \to \frac{1}{\epsilon}$ as $\epsilon \to 0$, the far-field oscillations are very fast.

3 Analytical Method

In this section, we will construct the oscillatory tails and estimate their amplitude by transforming the problem into a Fourier domain and using a perturbation analysis in the Fourier domain as in Akylas & Yang [1]. The Fourier transform of equation (2.3) gives

$$\hat{\eta} = \frac{1}{\epsilon} f(\tilde{k}) \,\text{cosech}(\pi\tilde{k}/2\epsilon\gamma), \tag{3.1}$$

where $f(\tilde{k}) = 3\tilde{k} + \frac{15}{2}\tilde{k}^2 + \cdots$, and $\tilde{k} = k\epsilon$. Substituting equation (3.1) in the equation resulting from the Fourier transform of equation (2.2), we obtain to the leading order in ϵ the following Volterra integral equation for $f(\tilde{k})$:

$$\tilde{k}^2(\tilde{k}^2 - 1)f(\tilde{k}) \ + \ 2\int_0^{\tilde{k}} f(\tilde{l})f(\tilde{k} - \tilde{l})d\tilde{l} = 0. \tag{3.2}$$

The solution of equation (3.2) can be obtained in the form of a power series given by

$$f(\tilde{k}) = \sum_{m=0}^{\infty} b_m \tilde{k}^{2m+1}, \tag{3.3}$$

where the coefficients b_m satisfy the recurrence relation

$$-\frac{(2m-1)(2m+6)}{(2m+3)(2m+2)} b_m + b_{m-1} + 2\sum_{r=1}^{m-1} \frac{(2m-2r+1)!(2r+1)!}{(2m+3)!} b_r b_{m-r} = 0, \quad m \geq 2, \tag{3.4}$$

with $b_0 = 3$ and $b_1 = 15/2$. As $m \to \infty$, the non-linear term in equation (3.4) becomes less important. So, we obtain $b_m \approx b_{m-1} \approx K$ as $m \to \infty$ where K is a constant. The value of K can be obtained by evaluating the values of b_m from equation (3.4) up to some large values of m. It is found that $K = 29.96$. Thus, the series (3.3) is convergent for $|\tilde{k}| < 1$ and has pole singularities at $\tilde{k} = \pm 1$ with f given by

$$f(\tilde{k}) \ \approx \ -\frac{K}{2(\tilde{k} \mp 1)} \text{ as } \tilde{k} \to \pm 1. \tag{3.5}$$

In view of equation (3.1) and (3.5), $\hat{\eta}$ has pole singularities at $k = \pm 1/\epsilon$ and is given by

$$\hat{\eta} \ \approx \ \mp \frac{K}{\epsilon^2(k \mp 1/\epsilon)}\, e^{-\pi/2\gamma\epsilon} \text{ as } k \to \pm 1/\epsilon. \tag{3.6}$$

Taking the inverse Fourier transform of equations (3.1) and (3.6) and using the residue theorem, we obtain

$$\eta(x) = \mathrm{PV}\int_{-\infty}^{\infty} \hat{\eta}(k)e^{ikx}dk \;+\; \frac{2\pi K}{\epsilon^2} e^{-\left(\frac{\pi}{2\gamma\epsilon}\right)} \sin\left(\frac{|x|}{\epsilon}\right). \tag{3.7}$$

The first term denotes the Cauchy principal value (PV) integral which corresponds to the asymptotic solution (2.3). The second term quantifies the oscillatory behavior (2.5) of the solution in far-field $x \to \pm\infty$.

4 Numerical Method

In this section, we solve equation (2.2) numerically using a pseudo-spectral method. The spectral basis functions are chosen suitably as a combination of rational Chebychev and radiation basis functions to get the correct solitary wave behavior (2.3) at the core (near $x = 0$) and oscillatory behavior (2.5) in the far-field (as $x \to \pm\infty$). Since the method is described in detail in Boyd [2], we only give a brief outline here.

Suppose $\eta^{(i)}(x)$ is the solution at ith iterate and $\delta\eta^{(i)}(x)$ is a correction to $\eta^{(i)}(x)$ such that $\eta(x) = \eta^{(i)}(x) + \delta\eta^{(i)}(x)$ satisfies equation (2.2). Substituting it in equation (2.2) and linearizing, we obtain the following linear inhomogeneous ODE (known as Newton-Kantorovich equation) for $\delta\eta^{(i)}(x)$:

$$\Big((1-c^2) + 2\eta^{(i)}\Big)\delta\eta^{(i)} + \delta\eta^{(i)}_{xx} + \epsilon^2\delta\eta^{(i)}_{xxxx} = -\Big[\Big((1-c^2) + \eta^{(i)}\Big)\eta^{(i)} + \eta^{(i)}_{xx} + \epsilon^2\eta^{(i)}_{xxxx}\Big]. \tag{4.1}$$

This iteration procedure is repeated until the correction $\delta\eta^{(i)}(x)$ becomes negligibly small. The solitary wave solution (2.3) is taken as the initial guess for small values ϵ. For large values of ϵ, the method of continuation (Boyd [2]) is used to find a suitable initial guess Now, if we write the solution at ith iterate as

$$\eta^{(i)}(x) \;=\; \sum_{n=1}^{N-1} a_n^{(i)}\Phi_n(x) \;+\; \Phi_{rad}(x; A^{(i)}), \tag{4.2}$$

then the correction to the solution at ith iterate will be given by

$$\delta\eta^{(i)}(x) \;\approx\; \sum_{n=1}^{N-1} \delta a_n^{(i)}\Phi_n(x) \;+\; \delta A^{(i)}\Phi_{rad,A}(x; A^{(i)}). \tag{4.3}$$

The amplitude A of the tail oscillations is obtained as a part of the solution along with the spectral coefficients a_n, $n = 1, 2, \cdots, N-1$. The spectral basis functions $\Phi_n(x)$ and $\Phi_{rad}(x; A)$ are constructed as

$$\Phi_n(x) \;=\; TB_{2n}(x) - 1, \quad \text{and} \quad \Phi_{rad}(x; A) \;=\; H(x)\eta_{cn}(x; A) \;+\; H(-x)\eta_{cn}(-x; A), \tag{4.4}$$

where $TB_{2n}(x) = \cos\left[2n\cot^{-1}(x/L)\right]$, $L = 2/\gamma$ are the rational Chebychev functions. Since $TB_{2n}(x)$ are even and asymptote to 1 as $x \to \pm\infty$, the basis functions $\Phi_n(x)$ are even and decay down to zero in the far field. Thus, the series $\sum_n^{N-1} a_n\Phi_n(x)$ gives the right behavior of the symmetric core solitary wave with peak at $x = 0$. The oscillatory behavior of the solution at tail ends is visualized by the radiation basis function $\Phi_{rad}(x; A)$ through its dependence on the cnoidal function $\eta_{cn}(x, A)$ which is given by

$$\eta_{cn}(x; A) \;=\; A\sin\Big[\frac{q}{\epsilon}(x+\phi)\Big] \;+\; A^2\Big[C_1 + C_2\cos\Big[\frac{2q}{\epsilon}(x+\phi)\Big]\Big] \;+\; A^3C_3\sin\Big[\frac{3q}{\epsilon}(x+\phi)\Big] + O(A^4), \tag{4.5}$$

where $q = q_0 + A^2q_2 + O(A^4)$, $q_0 = \left(1 + 4\epsilon^2\gamma^2\right)^{1/2}$, $q_2 = \epsilon^4(C_2 - 2C_1)/(2q_0^3 - q_0)$, $C_1 = \epsilon^2/2(q_0^2 - q_0^2)$, $C_2 = \epsilon^2/(30q_0^4 - 6q_0^2)$, and $C_3 = \epsilon^4/48(50q_0^8 - 15q_0^4 + q_0^2)$. The $\phi = 0$ corresponds to the case in which both core solitary wave and oscillatory tails are in phase. The smoothed step function $H(x)$ is suitably chosen in order to have the asymptotic behavior $H(x) \sim 1$ as $x \to \infty$ and $H(x) \sim 0$ as $x \to -\infty$. For simplicity, we choose

$$H(x) = \frac{1}{2}\Big[1 + \tanh\big(\gamma(x+\phi)\big)\Big]. \tag{4.6}$$

Since we are interested in obtaining symmetric solutions of equation (2.2) with peak at $x = 0$ and phase shift constant $\phi = 0$, we choose the N spectral grid (collocation) points all on positive real axis given by

$$x_n = L\cot\left[(2n-1)\pi/4N\right], \quad n = 1, 2, \cdots, N. \tag{4.7}$$

Substituting the spectral series (4.3) into the Newton-Kantorovich equation (4.1) and requiring that the residual vanish at N collocation points defined above, we obtain the matrix equation $JE = F$. Here $E = [\delta a_1^{(i)}, \delta a_2^{(i)}, \cdots, \delta a_{N-1}^{(i)}, \delta A^{(i)}]^T$, $F = [F_1^{(i)}, F_2^{(i)}, \cdots, F_N^{(i)}]^T$ and $J = [J_{nj}^{(i)}]$ is the Jacobian matrix of the resulting system of equations. Explicitly $J_{nj}^{(i)}$ and $F_n^{(i)}$ for $n = 1, 2, \cdots, N$ are expressed as

$$J_{nj}^{(i)} = \begin{cases} \left[\left((1-c^2) + 2\eta^{(i)}\right)\phi_j + \phi_{j,xx} + \epsilon^2\phi_{j,xxxx}\right]\Big|_{x=x_n} & \text{for} \quad j = 1, 2, \cdots, N-1, \\ \left[\left((1-c^2) + 2\eta^{(i)}\right)\phi_{rad,A} + \phi_{rad,Axx} + \epsilon^2\phi_{rad,Axxxx}\right]\Big|_{x=x_n} & \text{for} \quad j = N, \end{cases} \tag{4.8}$$

and

$$F_n^{(i)} = \left[\left((1-c^2) + \eta^{(i)}\right)\eta^{(i)} + \eta_{xx}^{(i)} + \epsilon^2\eta_{xxxx}^{(i)}\right]\Big|_{x=x_n}. \tag{4.9}$$

The matrix equation $JE = F$ is solved for the unknown vector E using Gaussian Elimination with partial pivoting. The iteration procedure is continued until the L_∞ norm of the vector E is negligibly small.

5 Numerical Results

The analytical estimate of amplitude A of the oscillatory tails for different values of the perturbation parameter ϵ^2 and phase speed c is shown in Table 1. It is observed that, the amplitude A of the oscillatory tails is exponentially small as compared to the amplitude of the core which is approximately equal to $6\gamma^2$ or $1.5(c^2-1)$. Also it decreases exponentially fast as the value of ϵ and c decreases.

Table 1: The analytical estimate of amplitude A of the oscillatory tails for different values of ϵ^2 and c.

$\epsilon^2 \backslash c$	1.05	1.10	1.15	1.20	1.25
0.0025	0.430405E-80	0.206454E-54	0.640770E-43	0.520739E-36	0.293631E-31
0.0100	0.433390E-38	0.295349E-25	0.162456E-19	0.458033E-16	0.107685E-13
0.0225	0.301413E-24	0.106437E-15	0.704676E-12	0.138968E-09	0.523737E-08
0.0400	0.209484E-17	0.529839E-11	0.383407E-08	0.199358E-06	0.300032E-05
0.0625	0.238900E-13	0.310892E-08	0.594467E-06	0.138618E-04	0.120067E-03
0.0900	0.112281E-10	0.201547E-06	0.158387E-04	0.216081E-03	0.129365E-02
0.1225	0.861493E-09	0.374824E-05	0.155859E-03	0.144863E-02	0.665788E-02
0.1600	0.214244E-07	0.321447E-04	0.828458E-03	0.577118E-02	0.217482E-01
0.2025	0.252448E-06	0.165261E-03	0.293416E-02	0.163294E-01	0.527304E-01
0.2500	0.176862E-05	0.595707E-03	0.784714E-02	0.364899E-01	0.104153E+00

The numerical results are obtained for phase shift constant $\phi = 0$ at various values of the perturbation parameter ϵ^2 and phase speed c. However, the results are presented with respect to a combined (group) parameter $\epsilon^2(c^2-1)$. The numerically computed amplitude of the oscillatory tails is compared with the corresponding analytical estimate in the left side graph of Figure 1. This graph shows the variation of $2\epsilon^2 A$ with the group parameter $\epsilon^2(c^2-1)$. It is observed that the numerically computed amplitude of the far-field oscillations agrees well with the analytical estimate for small values of $\epsilon^2(c^2-1)$. However, for larger values of $\epsilon^2(c^2-1)$, there is a small discrepancy between the two estimates which is expected since the analytical estimate is based on the asymptotic analysis for $\epsilon \ll 1$. Also, it is to be noted that the amplitude decreases exponentially fast as the value of $\epsilon^2(c^2-1)$ decreases.

The right side graph of Figure 1 shows the numerically computed symmetric weakly non-local solitary wave solution of the sixth-order Boussinesq equation (1.1) for $\epsilon^2(c^2-1) = 0.1$. For this moderate value of $\epsilon^2(c^2-1)$, the oscillatory tail is clearly visible. However, the oscillatory tail is exponentially small in comparison to the amplitude of the core solitary wave which is centered on the origin $x = 0$. The core in the neighborhood of $x = 0$ is best described by the solution (2.3). As the value of $\epsilon^2(c^2-1)$ decreases, the oscillatory tails decrease in amplitude and eventually collapse into the local solitary wave solution of the fourth-order Boussinesq equation.

6 Concluding Remarks

In Daripa & Hua [3], a sixth-order Boussinesq equation was introduced as a dispersive regularization of the ill-posed fourth-order Boussinesq equation. We analyzed this equation to find the traveling wave solutions.

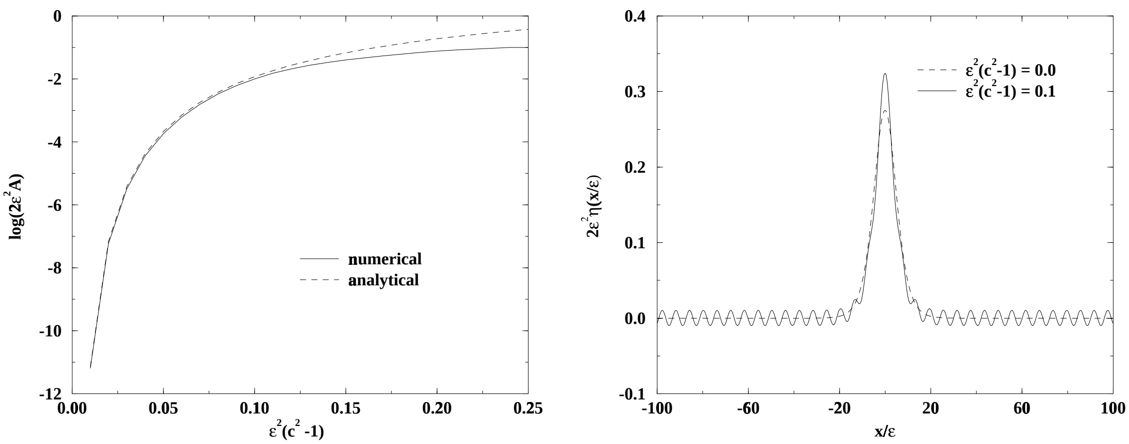

Figure 1: Left Graph: Comparison of the numerically computed amplitude (solid lines) of the oscillatory tails with that of the analytical estimate (dashed lines). Right Graph: Plots for traveling wave solutions of the sixth-order Boussinesq equation for $\epsilon^2(c^2-1) = 0.1$ and $\epsilon^2(c^2-1) = 0.0$.

On the basis of far-field analyses and heuristic arguments, we established that the traveling wave solutions of this equation do not vanish in the far-field. Instead, we showed that this equation admit weakly non-local solitary wave solutions, similar to that of the fifth-order KDV equation (e.g., Akylas & Yang [1], Boyd [2], Grimshaw & Joshi [6], Pomeau et al. [8]). This behavior is also consistent with the initial value calculation of Daripa & Hua [3]. Here we have also estimated the amplitude of the far-field oscillations both analytically and numerically and found them in excellent agreement with each other for small values of the perturbation parameter ϵ.

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