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Higher-order Boussinesq equations for two-way propagation of shallow water waves

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Abstract

Standard perturbation methods are applied to Euler's equations of motion governing the capillary-gravity shallow water waves to derive a general higher-order Boussinesq equation involving the small-amplitude parameter, $\alpha = a/h_0$, and long-wavelength parameter, $\beta = (h_0/l)^2$, where a and l are the actual amplitude and wavelength of the surface wave, and h_0 is the height of the undisturbed water surface from the flat bottom topography. This equation is also characterized by the surface tension parameter, namely the Bond number $\tau = \Gamma/\rho g h_0^2$, where Γ is the surface tension coefficient, ρ is the density of water, and g is the acceleration due to gravity.

The general Boussinesq equation involving the above three parameters is used to recover the classical model equations of Boussinesq type under appropriate scaling in two specific cases: (1) $|\frac{1}{3} - \tau| \gg \beta$, and (2) $|\frac{1}{3} - \tau| = \mathrm{O}(\beta)$. Case 1 leads to the classical (ill-posed and well-posed) fourth-order Boussinesq equations whose dispersive terms vanish at $\tau = \frac{1}{3}$. Case 2 leads to a sixth-order Boussinesq equation, which was originally introduced on a heuristic ground by Daripa and Hua [P. Daripa, W. Hua, A numerical method for solving an illposed Boussinesq equation arising in water waves and nonlinear lattices, Appl. Math. Comput. 101 (1999) 159–207] as a dispersive regularization of the ill-posed fourth-order Boussinesq equation. The relationship between the sixth-order Boussinesq equation and fifth-order KdV equation is also established in the limiting cases of the two small parameters α and β.

Keywords: Shallow water waves; Two-way wave propagation; Boussinesq equations; Solitary waves; Perturbation analysis

1. Introduction

Theoretical models of shallow water waves are often derived under application driven assumptions facilitating analysis and numerical computation. The hope is that these models are accurate enough for the intended applications. There are numerous models because no single model can capture all the phenomena associated with shallow water waves. For example, the family of KdV equations describes the uni-directional propagation of shallow water waves, whereas the family of Boussinesq equations describes the bi-directional propagation of such waves (e.g., see Bona et al. [1], Daripa and Dash [2], Johnson [3] and Whitham [4]). Each model within each family has its own range of applicability. For example, with surface tension effects included, the third-order KdV and fourth-order Boussinesq

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equations are appropriate for Bond number greater than 1/3 (e.g., see Amick and Kirchgässner [5] and Hunter and Vanden-Broeck [6]), whereas the fifth-order KdV and sixth-order Boussinesq equations are appropriate for Bond number less than but very close to 1/3 (e.g., see Dash and Daripa [7] and Hunter and Scheurle [8]).

It is well known that the velocity field in the shallow water is actually more complicated than these models would seem to indicate. This is not so surprising as these models are valid when the "small-amplitude" and "long-wavelength" parameters bear a certain relationship as they approach zero. These restrictions are too rigid and are very unlikely to hold in general in shallow water for arbitrary values of these parameters, however small these might be. More appropriate models for shallow water waves that can more accurately predict actual velocity field and other associated quantities can be obtained by incorporating the effect of each of these two parameters. In fact, this strategy can be used in a straight-forward manner to derive the shallow water wave models that would lead to the family of KdV and Boussinesq equations.

The classical fourth-order Boussinesq equation $\eta_{tt} = \eta_{xx} + (n^2)_{xx} + \eta_{xxxx}$, discussed in Daripa and Hua [9], possesses solitary wave solutions. However, as an initial value problem (IVP), it suffers from severe short-wave instability. The linearized version of this equation admits solutions in the form $e^{\sigma t + ikx}$ with short-wave instability $\sigma \approx k^2$ as $k \to \infty$. A consequence of this short-wave instability is the possible non-existence of classical solutions to this equation for arbitrary initial data except for some isolated solutions such as the classical solitary wave solutions. Another consequence of this short-wave instability is difficulty in numerically constructing good approximate solutions of even known solutions (see Daripa [10] and Daripa and Hua [9]). These facts seriously cast doubts on the real utility of the classical (ill-posed) fourth-order Boussinesq equation in spite of its frequent appearance in most books (e.g., Johnson [3] and Whitham [4]) on non-linear waves and water waves as a model equation for bi-directional propagation of small-amplitude long waves.

Due to the severe ill-posedness of the classical fourth-order Boussinesq equation and the associated mathematical and numerical difficulties, there has been a considerable interest in devising equivalent Hamiltonian and well-posed model equations for bi-directional wave propagation (e.g., see Bona and Chen [11], Bona et al. [1], Chen [12], Daripa and Dash [2] and Olver [13,14]). Based on a general theory of non-canonical perturbations of Hamiltonian systems, Olver [13,14] derived some new Hamiltonian model equations for both uni- and bi-directional propagation of small-amplitude long waves on the surface of shallow water. Later, through some detailed analytical studies, Kichenassamy [15] and Kichenassamy and Olver [16] studied various higher-order model equations for water waves, including the well-known fifth-order KdV equation, and investigated the issue of existence of solitary wave solutions for those equations. Berger and Milewski [17] numerically studied the generalized Benney–Luke equation (Milewski [18]) to study the generation and evolution of lump solitary waves in surface tension dominated flows. Bona et al. [1] derived a number of variants of the classical Boussinesq system for such bi-directional wave propagation problems and presented their higher-order generalizations, including their relevance to experiments and observations (also see Bona and Chen [11], Chen [12], and the references their in). However, the effect of surface tension was not considered in their studies.

Ill-posed interfacial model equations are often regularized by adding the effect of surface tension (e.g., see Daripa and Hua [9] and Joseph et al. [19]). When the effect of surface tension is included, solutions to the water wave problems are characterized by the Bond number $\tau = \Gamma/\rho g h_0^2$, in addition to the amplitude parameter $\alpha = a/h_0$ and the wavelength parameter $\beta = (h_0/l)^2$, where Γ is the surface tension coefficient, ρ is the density of water, g is the acceleration due to gravity, h_0 is the height of the undisturbed water surface, a is the amplitude of the surface wave, and l is the wavelength of the surface wave. Even though the surface tension itself may not be that important for small amplitude long waves, its inclusion, however small, in deriving model equations for propagation of such waves may be important, in particular, in non-linear models that may otherwise generate dangerous short waves similar to the ones that arise in classical fourth-order Boussinesq equation. In some cases, the inclusion of surface tension effect leads to higher-order model equations such as the fifth-order KdV equation (see Hunter and Scheurle [8]) and the sixth-order Boussinesq equation (see Daripa and Hua [9], Daripa and Dash [2] and Dash and Daripa [7]). The equations considered in these papers are physically relevant model equations for shallow water waves in the limit $\tau \uparrow 1/3$ (i.e., when the Bond number τ is less than but very close to 1/3). It has been proved that these equations do not possess classical local solitary wave solutions, but admit weakly non-local solitary wave solutions characterized by oscillatory tails in the far-field (see Akylas and Yang [20], Boyd [21], Daripa and Hua [9], Dash and Daripa [7], Grimshaw and Joshi [22], Hunter and Scheurle [8], Kichenassamy [15], Kichenassamy and Olver [16] and Pomeau and Ramani [23]).

In this paper, we are concerned with general higher-order model equations of Boussinesq type containing both the amplitude and wavelength parameters in addition to the Bond number (surface tension parameter). Such a model will have extended range of applicability than the ones with only one parameter. One obvious situation where two-way propagation is desirable is when the flow is bounded by walls. There is no sensible way to study wall reflection in either third-order KdV equation or its fifth-order generalization. However, the higher-order Boussinesq equations circumvent this problem and allow the possibility of such studies. Furthermore, as mentioned before, inclusion of the effect of surface tension makes some, not all, shallow water wave models physically more relevant and numerically well-posed for small values of surface tension such as Eq. (3.39) derived later in Section 3 (see also Daripa [10], Daripa and Hua [9], Daripa and Dash [2], and Dash and Daripa [7]).

In Section 2, the Euler's equations describing the dynamics of capillary-gravity water waves in two-dimensions are considered in the limits of small-amplitude and long-wavelength with appropriate boundary conditions. In Section 3, using a double-series perturbation analysis, a general higher-order equation of Boussinesq type is derived containing both the small-amplitude and long-wavelength parameters (α and β). In Section 4, the fourth-order and sixth-order Boussinesq equations are recovered when these two parameters bear certain relationships as they approach zero. Some remarks on the fourth-order and sixth-order Boussinesq equations are also made and their connection to the well-known fifth-order KdV equation is established. Solutions of the equations are briefly introduced in Section 5. We have summarized our results and discussed their relevance in Section 6. In Section 7, the concluding remarks are made.

2. Formulation of the problem

Let $z=0$ represent the bottom topography and $z=h(x,t)=h_0+a\eta(x,t)$ represent the free water surface, where h_0 is the height of the undisturbed water surface, a is the amplitude of the surface wave, and $\eta(x,t)$ is the free surface elevation from its undisturbed location. Let (u,w) represent the velocity field in (x,z) co-ordinate. We use the following non-dimensionalization

$$\begin{aligned} &x\to lx,\quad z\to h_0 z,\quad t\to\frac{l}{\sqrt{gh_0}}t,\quad u\to\frac{a}{h_0}\sqrt{gh_0}u,\\ &w\to\left(\frac{a}{h_0}\right)\left(\frac{h_0}{l}\right)\sqrt{gh_0}w,\quad p\to p_a+\rho g(h_0-z)+\frac{a}{h_0}(\rho g h_0)p, \end{aligned}\tag{2.1}$$

where l is the wavelength of the surface wave, g is the acceleration due to gravity, ρ is the density of the fluid, p is the pressure field, and p_a is the atmospheric pressure.

In non-dimensional form, Euler's equations of motion governing the capillary-gravity shallow water waves (see Johnson [3]) are given by

$$\begin{aligned} &u_t+\alpha(uu_x+wu_z)=-p_x,\\ &\beta\big[w_t+\alpha(uw_x+ww_z)\big]=-p_z,\\ &u_x+w_z=0. \end{aligned}\tag{2.2}$$

The corresponding kinematic and dynamic boundary conditions are given by

$$\begin{aligned} &w=0\quad\text{at } z=0,\\ &w=\eta_t+\alpha u\eta_x\quad\text{at } z=1+\alpha\eta,\\ &p=\eta-\beta\tau\frac{\eta_{xx}}{[1+\alpha^2\beta\eta_x^2]^{3/2}}\quad\text{at } z=1+\alpha\eta. \end{aligned}\tag{2.3}$$

Here $\alpha=a/h_0$ (amplitude parameter), $\beta=(h_0/l)^2$ (wavelength parameter) and $\tau=\Gamma/\rho g h_0^2$ (Bond number), where Γ is the surface tension coefficient. It is known that full non-linear water wave equations (2.2) and (2.3) are Hamiltonian and possess energy conserving functional.

The linearized version of the above equations admits solutions for η in the form of $A\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}$ provided the following dispersion relation holds (see Whitham [4])

$$\omega^2=\frac{c_0^2}{h_0^2}\big[\big(1+\tau k^2h_0^2\big)kh_0\tanh(kh_0)\big],\tag{2.4}$$

where $c_0=\sqrt{gh_0}$. In the long wavelength limit (i.e., $kh_0\ll 1$), we have

$$\omega^2=c_0^2k^2\left[1-\left(\frac{1}{3}-\tau\right)k^2h_0^2+\frac{1}{3}\left(\frac{2}{5}-\tau\right)k^4h_0^4-\frac{2}{15}\left(\frac{17}{42}-\tau\right)k^6h_0^6+\frac{17}{315}\left(\frac{62}{153}-\tau\right)k^8h_0^8-\cdots\right]. \tag{2.5}$$

This indicates that the leading order dispersion term in the equation for η is of the order $(\frac{1}{3}-\tau)k^2h_0^2$, or equivalently of the order of $(\frac{1}{3}-\tau)\beta$. Therefore, the leading order dispersion term is of $\mathrm{O}(\beta)$ if $|\frac{1}{3}-\tau|=\mathrm{O}(1)$ and $\mathrm{O}(\beta^2)$ if $|\frac{1}{3}-\tau|=\mathrm{O}(\beta)$. However, the non-linear term is always of order α irrespective of the value of Bond number τ. Therefore, a balance between the non-linear and dispersive effects (which is necessary to model solitary waves) requires that $\alpha=\mathrm{O}(\beta)$ when $|\frac{1}{3}-\tau|=\mathrm{O}(1)$ and $\alpha=\mathrm{O}(\beta^2)$ when $|\frac{1}{3}-\tau|=\mathrm{O}(\beta)$. Thus, for Bond number very close 1/3 (i.e., for $\tau\uparrow 1/3$ or $\tau\downarrow 1/3$), we need to have

$$\left|\frac{1}{3}-\tau\right|=K_1\beta \quad \text{and} \quad \alpha=K_2\beta^2 \quad \text{as } \beta\to 0, \tag{2.6}$$

with non-zero positive constants K_1 and K_2 (fixed). This relation will be used later in establishing the physical relevance of the sixth-order Boussinesq equation.

Before proceeding for analysis, we rewrite the boundary conditions (2.3) by expressing these at $z=0$ and $z=1$ through a Taylor series expansion for u, w and p. So, we have

$$\begin{aligned}
&w=0 \quad \text{at } z=0,\\
&w+\alpha\eta w_z+\frac{\alpha^2\eta^2}{2}w_{zz}=\eta_t+\alpha\eta_x(u+\alpha\eta u_z)+\mathrm{O}(\alpha^3) \quad \text{at } z=1,\\
&p+\alpha\eta p_z+\frac{\alpha^2\eta^2}{2}p_{zz}=\eta-\beta\tau\eta_{xx}+\mathrm{O}(\alpha^3,\alpha^2\beta^2) \quad \text{at } z=1.
\end{aligned} \tag{2.7}$$

In the section below, we derive a general higher-order model equation of Boussinesq type for η from governing equations (2.2) and boundary conditions (2.7) by suitably eliminating u, w and p through a double-series perturbation analysis.

3. Derivation of a general higher-order Boussinesq equation

We express the solutions of the unknown variables $q=(u,w,p,\eta)$ in the problem through a double power series of the form

$$q=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha^i\beta^jq_{ij}=q_{00}+\alpha q_{10}+\beta q_{01}+\alpha^2q_{20}+\beta^2q_{02}+\alpha\beta q_{11}+\cdots. \tag{3.1}$$

Upon substituting expansions (3.1) for u, w, p and η into the governing equations (2.2) and boundary conditions (2.7), we obtain the following equations and boundary conditions of various orders as the coefficients of $\alpha^i\beta^j$, $i=0,1,\ldots$, $j=0,1,\ldots$

• *Equations of* $\mathrm{O}(1)$:

$$u_{00t}=-p_{00x}, \qquad p_{00z}=0, \qquad u_{00x}+w_{00z}=0, \tag{3.2}$$

with

$$w_{00}=0 \quad \text{at } z=0, \qquad w_{00}=\eta_{00t} \quad \text{at } z=1, \qquad p_{00}=\eta_{00} \quad \text{at } z=1. \tag{3.3}$$

• *Equations of* $\mathrm{O}(\alpha)$:

$$u_{10t}+u_{00}u_{00x}+w_{00}u_{00z}=-p_{10x}, \qquad p_{10z}=0, \qquad u_{10x}+w_{10z}=0, \tag{3.4}$$

with

$$\begin{aligned}
&w_{10}=0 \quad \text{at } z=0, \qquad w_{10}+\eta_{00}w_{00z}=\eta_{10t}+\eta_{00x}u_{00} \quad \text{at } z=1,\\
&p_{10}+\eta_{00}p_{00z}=\eta_{10} \quad \text{at } z=1.
\end{aligned} \tag{3.5}$$

• *Equations of* $\mathrm{O}(\beta)$:

$$u_{01t} = -p_{01x}, \qquad w_{00t} = -p_{01z}, \qquad u_{01x} + w_{01z} = 0, \tag{3.6}$$

with

$$w_{01} = 0 \quad \text{at } z = 0, \qquad w_{01} = \eta_{01t} \quad \text{at } z = 1, \qquad p_{01} = \eta_{01} - \tau\eta_{00xx} \quad \text{at } z = 1. \tag{3.7}$$

• *Equations of* $\mathrm{O}(\alpha^2)$:

$$u_{20t} + (u_{00}u_{10})_x + w_{00}u_{10z} + w_{10}u_{00z} = -p_{20x}, \qquad p_{20z} = 0, \qquad u_{20x} + w_{20z} = 0, \tag{3.8}$$

with

$$\begin{aligned} &w_{20} = 0 \quad \text{at } z = 0,\\ &w_{20} + \eta_{00}w_{10z} + \eta_{10}w_{00z} + \frac{\eta_{00}^2}{2}w_{00zz} = \eta_{20t} + \eta_{00x}u_{10} + \eta_{10x}u_{00} + \eta_{00x}\eta_{00}u_{00z} \quad \text{at } z = 1,\\ &p_{20} + \eta_{00}p_{10z} + \eta_{10}p_{00z} + \frac{\eta_{00}^2}{2}p_{00z} = \eta_{20} \quad \text{at } z = 1. \end{aligned} \tag{3.9}$$

• *Equations of* $\mathrm{O}(\beta^2)$:

$$u_{02t} = -p_{02x}, \qquad w_{01t} = -p_{02z}, \qquad u_{02x} + w_{02z} = 0, \tag{3.10}$$

with

$$w_{02} = 0 \quad \text{at } z = 0, \qquad w_{02} = \eta_{02t} \quad \text{at } z = 1, \qquad p_{02} = \eta_{02} - \tau\eta_{01xx} \quad \text{at } z = 1. \tag{3.11}$$

• *Equations of* $\mathrm{O}(\alpha\beta)$:

$$\begin{aligned} &u_{11t} + (u_{00}u_{01})_x + w_{00}u_{01z} + w_{01}u_{00z} = -p_{11x}, \qquad w_{10t} + u_{00}w_{00x} + w_{00}w_{00z} = -p_{11z},\\ &u_{11x} + w_{11z} = 0, \end{aligned} \tag{3.12}$$

with

$$\begin{aligned} &w_{11} = 0 \quad \text{at } z = 0, \qquad w_{11} + \eta_{00}w_{01z} + \eta_{01}w_{00z} = \eta_{11t} + \eta_{00x}u_{01} + \eta_{01x}u_{00} \quad \text{at } z = 1,\\ &p_{11} + \eta_{00}p_{01z} + \eta_{01}p_{00z} = \eta_{11} - \tau\eta_{10xx} \quad \text{at } z = 1. \end{aligned} \tag{3.13}$$

Below we derive canonical equations for η_{00}, η_{10}, η_{01}, η_{20}, η_{02} and η_{11} from the above set of equations by eliminating the other variables, namely u, w and p. These equations are then combined according to the perturbation series (3.1) to obtain the appropriate Boussinesq equation for η.

• *Equation for* $\eta_{00}(x,t)$: It is easy to see from Eqs. (3.2) and (3.3) that

$$p_{00} = \eta_{00}, \quad u_{00t} = -\eta_{00x}, \quad w_{00} = -zu_{00x}, \quad u_{00x} = -\eta_{00t}, \tag{3.14}$$

and hence we obtain the equation for η_{00} as

$$\eta_{00tt} - \eta_{00xx} = 0. \tag{3.15}$$

Therefore, the solution η_{00} will be of traveling wave form $E_{00}(x-t) + F_{00}(x+t)$ for some arbitrary functions E_{00} and F_{00}.

• *Equation for* $\eta_{10}(x,t)$: From Eq. (3.4b) and condition (3.5c), we have

$$p_{10} = \eta_{10}. \tag{3.16}$$

By Eq. (3.4b), we mean the second subequation in (3.4) and by Eq. (3.5c) we mean the third equation in (3.5). Similar convention has been used below. Substituting Eq. (3.16) in Eq. (3.4a) and noting that u_{00} is independent of z, we obtain

$$u_{10t} = -\eta_{10x} - \frac{1}{2}\left(u_{00}^2\right)_x. \tag{3.17}$$

Thus u_{10} is also independent of z, and therefore, direct integration of Eq. (3.4c) with the help of condition (3.5a) gives

$$w_{10} = -zu_{10x}. \tag{3.18}$$

By matching Eq. (3.18) with condition (3.5b) and then using Eq. (3.14), we get

$$u_{10x} = -\eta_{10t} - (\eta_{00}u_{00})_x. \tag{3.19}$$

Eqs. (3.17) and (3.19) with the help of Eq. (3.14) now give

$$\eta_{10tt} - \eta_{10xx} - \left[\frac{1}{2}\eta_{00}^2 + u_{00}^2\right]_{xx} = 0. \tag{3.20}$$

Using $u_{00} = -\int_{-\infty}^{x} \eta_{00t}\,\mathrm{d}x$ from Eq. (3.14), we obtain for η_{10} the following equation.

$$\eta_{10tt} - \eta_{10xx} - \left[\frac{1}{2}\eta_{00}^2 + \left(\int_{-\infty}^{x} \eta_{00t}\,\mathrm{d}x\right)^2\right]_{xx} = 0. \tag{3.21}$$

Therefore, solution η_{10} will be of traveling wave form $E_{10}(x-t) + F_{10}(x+t) + tG_{10}(x-t) + tH_{10}(x+t)$ for some arbitrary functions E_{10} and F_{10}. The functions G_{10} and H_{10} are dependent on E_{00} and F_{00}.

• *Equations for* $\eta_{01}(x,t)$: Using Eq. (3.14) in Eq. (3.6b) and integrating the resulting equation with the help of condition (3.7c), we obtain

$$p_{01} = \left[\eta_{01} + \left(\frac{1}{2} - \tau\right)\eta_{00xx}\right] - \frac{1}{2}\eta_{00xx}z^2. \tag{3.22}$$

Eqs. (3.6a,c) and (3.22) then give

$$w_{01zt} = -u_{01xt} = p_{01xx} = \left[\eta_{01xx} + \left(\frac{1}{2} - \tau\right)\eta_{00xxxx}\right] - \frac{1}{2}\eta_{00xxxx}z^2. \tag{3.23}$$

On integrating Eq. (3.23) with the help of condition (3.7a) and then matching its value at $z = 1$ with the t-derivative of condition (3.7b), we obtain the equation for η_{01} as

$$\eta_{01tt} - \eta_{01xx} - \left(\frac{1}{3} - \tau\right)\eta_{00xxxx} = 0. \tag{3.24}$$

Therefore, the solution of η_{01} will be of traveling wave form $E_{01}(x-t) + F_{01}(x+t) + tG_{01}(x-t) + tH_{01}(x+t)$ for some arbitrary functions E_{01} and F_{01}. The functions G_{01} and H_{01} are dependent on E_{00} and F_{00}.

• *Equations for* $\eta_{20}(x,t)$: It is easy to see from Eq. (3.8b) and condition (3.9c) that

$$p_{20} = \eta_{20}. \tag{3.25}$$

Substituting Eq. (3.25) in Eq. (3.8a) and using the fact that u_{00} and u_{10} are independent of z, we obtain

$$u_{20t} = -\left[\eta_{20x} + (u_{00}u_{10})_x\right]. \tag{3.26}$$

Thus u_{20} is also independent of z, and therefore, direct integration of Eq. (3.8c) with the help of condition (3.9a) gives

$$w_{20} = -zu_{20x}. \tag{3.27}$$

By matching Eq. (3.27) with condition (3.9b) and then using Eq. (3.14), we get

$$u_{20x} = -\left[\eta_{20t} + (\eta_{00}u_{10})_x + (\eta_{10}u_{00})_x\right]. \tag{3.28}$$

Eqs. (3.26) and (3.28) with the help of Eqs. (3.14), (3.17) and (3.19) now give

$$\eta_{20tt} - \eta_{20xx} - (\eta_{00}\eta_{10})_{xx} - (2u_{00x}u_{10})_x + (2\eta_{10t}u_{00})_x + \left(\eta_{00x}u_{00}^2\right)_x = 0. \tag{3.29}$$

Using $u_{10} = -\int_{-\infty}^{x} \eta_{10t}\,\mathrm{d}x - \eta_{00}u_{00}$ (see (3.19)) in the above equation, we obtain

$$\eta_{20tt} - \eta_{20xx} - (\eta_{00}\eta_{10})_{xx} - 2\left[\int_{-\infty}^{x} \eta_{00t}\,\mathrm{d}x \int_{-\infty}^{x} \eta_{10t}\,\mathrm{d}x\right]_{xx} + \left[\eta_{00}\left(\int_{-\infty}^{x} \eta_{00t}\,\mathrm{d}x\right)^2\right]_{xx} = 0. \tag{3.30}$$

Therefore, the solution of η_{20} will be of traveling wave form $E_{20}(x-t)+F_{20}(x+t)+tG_{20}(x-t)+tH_{20}(x+t)+t^2P_{20}(x-t)+t^2Q_{20}(x+t)$ for some arbitrary functions E_{20} and F_{20}. The functions G_{20} and H_{20} are dependent on E_{00}, F_{00}, E_{10}, F_{10}, G_{10} and H_{10}, and the functions P_{20} and Q_{20} are dependent on E_{00} and F_{00}.

• *Equations for* $\eta_{02}(x,t)$: Using the expression for w_{01t} from Eq. (3.23) in Eq. (3.10b) and integrating the resulting equation with the help of condition (3.11c), we obtained

$$p_{02}=\left[\eta_{02}+\left(\frac{1}{2}-\tau\right)\eta_{01xx}+\frac{1}{2}\left(\frac{5}{12}-\tau\right)\eta_{00xxxx}\right]-\frac{1}{2}\left[\eta_{01xx}+\left(\frac{1}{2}-\tau\right)\eta_{00xxxx}\right]z^2+\frac{1}{24}\eta_{00xxxx}z^4. \tag{3.31}$$

Eqs. (3.10a,c) and (3.31) then give

$$w_{02zt}=-u_{02xt}=p_{02xx}=\left[\eta_{02xx}+\left(\frac{1}{2}-\tau\right)\eta_{01xxxx}+\frac{1}{2}\left(\frac{5}{12}-\tau\right)\eta_{00xxxxxx}\right]-\frac{1}{2}\left[\eta_{01xxxx}+\left(\frac{1}{2}-\tau\right)\eta_{00xxxxxx}\right]z^2+\frac{1}{24}\eta_{00xxxxxx}z^4. \tag{3.32}$$

On integrating Eq. (3.32) with the help of condition (3.11a) and then matching its value at $z=1$ with the t-derivative of condition (3.11b), we obtain the equation for η_{02} as

$$\eta_{02tt}-\eta_{02xx}-\left(\frac{1}{3}-\tau\right)\eta_{01xxxx}-\frac{1}{3}\left(\frac{2}{5}-\tau\right)\eta_{00xxxxxx}=0. \tag{3.33}$$

Therefore, the solution of η_{02} will be of traveling wave form $E_{02}(x-t)+F_{02}(x+t)+tG_{02}(x-t)+tH_{02}(x+t)+t^2P_{02}(x-t)+t^2Q_{02}(x+t)$ for some arbitrary functions E_{02} and F_{02}. The functions G_{02} and H_{02} are dependent on E_{00}, F_{00}, E_{01}, F_{01}, G_{01} and H_{01}, and the functions P_{02} and Q_{02} are dependent on E_{00} and F_{00}.

• *Equations for* $\eta_{11}(x,t)$: Using Eqs. (3.14), (3.17) and (3.18) in Eq. (3.12b) and integrating the resulting equation with the help of condition (3.13c), we obtain

$$p_{11}=\left[\eta_{11}+\left(\frac{1}{2}-\tau\right)\eta_{10xx}+\frac{1}{2}\eta_{00t}^2+\eta_{00}\eta_{00xx}\right]-\frac{1}{2}\left[\eta_{10xx}+\eta_{00t}^2\right]z^2. \tag{3.34}$$

Eqs. (3.12a,c), (3.14) and (3.34) then give

$$\begin{aligned}w_{11zt}&=-u_{11xt}=-p_{11xx}+(u_{00}u_{01})_{xx}+(w_{00}u_{01z})_x\\&=\left[(u_{00}u_{01})_{xx}+(w_{00}u_{01z})_x+\eta_{11xx}+\left(\frac{1}{2}-\tau\right)\eta_{10xxxx}+\frac{1}{2}\left(\eta_{00t}^2\right)_{xx}+(\eta_{00}\eta_{00xx})_{xx}\right]\\&\quad-\frac{1}{2}\left[\eta_{10xxxx}+\left(\eta_{00t}^2\right)_{xx}\right]z^2.\end{aligned} \tag{3.35}$$

On integrating Eq. (3.35) with the help of condition (3.13a) and then matching its value at $z=1$ with the t-derivative of condition (3.13b), we obtain

$$\int_0^1\left[(u_{00}u_{01})_{xx}+(w_{00}u_{01z})_x\right]\mathrm{d}x+\eta_{11xx}+\left(\frac{1}{3}-\tau\right)\eta_{10xxxx}+\frac{1}{3}\left(\eta_{00t}^2\right)_{xx}+(\eta_{00}\eta_{00xx})_{xx}=\eta_{11tt}+(\eta_{00}u_{01})_{xt}\big|_{z=1}+(\eta_{01}u_{00})_{xt}. \tag{3.36}$$

By the help of calculations based on O(1) and O(β) equations, it is easy to see that

$$\int_0^1\left[(u_{00}u_{01})_{xx}+(w_{00}u_{01z})_x\right]\mathrm{d}x=-u_{00x}\left[3w_{01}(1)-w_{01z}(1)\right]-u_{00}w_{01x}(1)-u_{00xx}\left[2\int_{-\infty}^{x}\eta_{01t}\,\mathrm{d}x+u_{01}(1)\right]. \tag{3.37}$$

So, Eq. (3.36) with the help of Eqs. (3.37), (3.2), (3.3), (3.6) and (3.7) reduces to the following equation for η_{11}:

$$\eta_{11tt} - \eta_{11xx} - (\eta_{00}\eta_{01})_{xx} - 2\left(\int_{-\infty}^{x} \eta_{00t}\,\mathrm{d}x \int_{-\infty}^{x} \eta_{01t}\,\mathrm{d}x\right)_{xx} - \left(\frac{1}{3} - \tau\right)\eta_{10xxxx}$$
$$- \frac{2}{3}\left(\eta_{00t}^2\right)_{xx} - (\eta_{00}\eta_{00xx})_{xx} + \tau(\eta_{00}\eta_{00xxx})_x = 0. \tag{3.38}$$

Therefore, $\eta_{11}(x,t)$ will be of traveling wave form $E_{11}(x-t) + F_{11}(x+t) + tG_{11}(x-t) + tH_{11}(x+t) + t^2P_{11}(x-t) + t^2Q_{11}(x+t)$ for some arbitrary functions E_{11} and F_{11}. The functions G_{11} and H_{11} are dependent on E_{00}, F_{00}, E_{01}, F_{01}, E_{10}, F_{10}, G_{01}, H_{01}, G_{10} and H_{10}, and the functions P_{11} and Q_{11} are dependent on E_{00} and F_{00}.

• *Equation for* $\eta(x,t)$: Combining Eqs. (3.15), (3.21), (3.24), (3.30), (3.33) and (3.38) according to the series expansion (3.1), we obtain the following equation for η accurate up to $\mathrm{O}(\alpha^2)$, $\mathrm{O}(\beta^2)$, and $\mathrm{O}(\alpha\beta)$.

$$\eta_{tt} - \eta_{xx} - \alpha\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t\,\mathrm{d}x\right)^2\right]_{xx} - \beta\left[\frac{1}{3} - \tau\right]\eta_{xxxx} + \alpha^2\left[\eta\left(\int_{-\infty}^{x} \eta_t\,\mathrm{d}x\right)^2\right]_{xx}$$
$$- \alpha\beta\left[\frac{2}{3}\left(\eta_t^2\right)_{xx} + (\eta\eta_{xx})_{xx} - \tau(\eta\eta_{xxx})_x\right] - \frac{\beta^2}{3}\left[\frac{2}{5} - \tau\right]\eta_{xxxxxx} = 0. \tag{3.39}$$

This is a general equation of Boussinesq type valid for small values of α and β. This equation includes the effect of surface tension through the bond number τ. It describes the bi-directional propagation of small amplitude long capillary-gravity waves on the surface of shallow water.

Since the solutions for the perturbed components η_{10}, η_{01}, η_{20}, η_{02} and η_{11} contain "secular terms" (the functions multiplied with t or t^2), these solutions will be unbounded as $t \to \infty$. Therefore, the perturbation series approximation (3.1) for η, and hence the general Boussinesq equation (3.39), will not be uniformly valid for all t. However, an examination of the secular terms indicates that the solution for η will be valid up to a time t for which both αt and βt are less than 1, that is, for all $0 \leqslant t < \epsilon^{-1}$ where $\epsilon = \max(\alpha, \beta)$.

Note that Eq. (3.39) is linearly well-posed for values of surface tension parameter $\tau < 2/5$ including $\tau = 0$!. At $\tau = 0$, this well-posed equation (3.39) remains a sixth-order equation. If Eq. (3.39) is truncated at $\mathrm{O}(\alpha)$ and $\mathrm{O}(\beta)$, then this equation is of fourth-order and is illposed. (This illposed fourth-order equation is not the same illposed fourth-order equation considered in Daripa and Hua [9].) Thus, extension of this fourth-order equation to sixth-order even at zero surface tension makes the problem well-posed. Also, note that the sixth-order equation (3.39) is not well-posed for $\tau > 2/5$. Thus excessive surface tension in this sixth-order model is destabilizing.

Lastly, it should be noted that Eq. (3.39) for zero or non-zero τ is not the same sixth-order equation (see (4.6) in the next section) that was originally introduced in [9]. In the next section, we show that the way to obtain the sixth-order equation (4.6) introduced in [9] is first to obtain (3.39) in two parameters α and β, then analyze this (see Section 4 – Case 2) in the limit $\alpha = \mathrm{O}(\beta^2)$ when $\tau \sim 1/3$. This leads to yet a different sixth-order equation (4.2), which has now only one parameter α and also includes surface tension effect implicitly through the constants K_1 and K_2. Lastly, this Eq. (4.2) is reduced (see Section 4.1) to the desired canonical form (4.6) through the non-linear transformation (4.3)–(4.4).

Related to Eq. (3.39), there are some important issues such as well-posedness of the equation, (non-)existence of Hamiltonian, (non-)existence of an energy-type of conservation property, etc. Investigation into definitive answers related to these issues will be taken up in the future and falls outside the scope of this paper. However, some remarks related to these issues can be made here. For example, due to the appearance of non-local terms in this equation and the secular terms as mentioned above, addressing the well-posedness of this equation in a definitive way is a non-trivial matter. However, if it is well-posed, its usefulness is restricted in time due to the above mentioned restricted length of time interval imposed by ϵ^{-1}, though this time interval can be large since ϵ is usually very small. It is very unlikely that this equation is Hamiltonian and thus makes it difficult to judge whether if it is conservative. These issues will be investigated in more detail in the future. However, we address these issues below for some reduced versions of this equation.

Remark 1. Due to conservation of mass, the integral $\int_{-\infty}^{\infty} h(x,t)\,\mathrm{d}x$ must be invariant in time and hence $\int_{-\infty}^{\infty} \eta(x,t)\,\mathrm{d}x = 0$ since we have taken $h(x,t) = h_0 + a\eta(x,t)$ (see the starting line of Section 2). Therefore,

$\int_{-\infty}^{x} \eta(x,t)\,\mathrm{d}x = \int_{\infty}^{x} \eta(x,t)\,\mathrm{d}x$. Same relation should also then hold true for $\eta_t(x,t)$. Thus the lower limit in the integral $\int^{x}$ involving either $\eta(x,t)$ or $\eta_t(x,t)$ that appears above in several equations can be either ∞ or $-\infty$.

4. Derivation of fourth-order and sixth-order Boussinesq equations

The fourth-order and sixth-order Boussinesq equations can be derived from Eq. (3.39) under appropriate scalings. It follows from Eq. (3.39) that the effect of non-linearity appears at $\mathrm{O}(\alpha)$, $\mathrm{O}(\alpha^2)$ and $\mathrm{O}(\alpha\beta)$ terms, whereas the effect of dispersion appears at $\mathrm{O}(\beta)$ and $\mathrm{O}(\beta^2)$ terms. The leading order dispersion term is $\beta(\frac{1}{3}-\tau)\eta_{xxxx}$. Therefore, it is important to consider two special cases: (1) $|\frac{1}{3}-\tau| \gg \beta$ and (2) $|\frac{1}{3}-\tau| = \mathrm{O}(\beta)$. The Case 1 leads to the fourth-order Boussinesq equation whose fourth-order dispersive term vanishes for $\tau = \frac{1}{3}$. This emphasizes the significance of the Case 2 which leads to the sixth-order Boussinesq equation. These are briefly presented below.

• *Case 1*: If $|\frac{1}{3}-\tau| \gg \beta$, i.e., $(\frac{1}{3}-\tau) = \pm K_1$ and $K_1 \gg \beta$, then a balance between the non-linearity and dispersion, which is necessary to model a solitary wave, requires $\alpha = \mathrm{O}(\beta)$ as $\beta \to 0$, i.e., $(\alpha/\beta) \to K_2 > 0$ as $\beta \to 0$. Then we have the Boussinesq equation (3.39) correct up to $\mathrm{O}(\alpha) = \mathrm{O}(\beta)$ as

$$\eta_{tt} - \eta_{xx} - \alpha\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t\,\mathrm{d}x\right)^2\right]_{xx} \mp \frac{K_1}{K_2}\alpha\,\eta_{xxxx} = 0. \tag{4.1}$$

This equation is appropriate for $0 \leqslant \tau \ll \frac{1}{3}$ when the fourth-order dispersive term is negative, and for $\frac{1}{3} \ll \tau$ when the fourth-order dispersive term is positive. This model is valid on a time-scale of the order $\mathrm{O}(1/\alpha) = \mathrm{O}(1/\beta)$. When this equation is restricted to a "submanifold" of approximately unidirectional waves, its yields KdV type equations which is Hamiltonian. It is very likely that Eq. (4.1), due to non-local terms, may not have energy-type conserving property and may not even have Hamiltonian formulation. However, we will see below (see Section 4.1) that rescaled versions of this equation involving new variables having physical interpretation is actually completely integrable and thus have many constants of motions, though none of these invariants of motion involving the new variable may correspond to actual energy-conserving property for our Eq. (4.1). These new variables with physical interpretations are introduced below in Section 4.1.

• *Case 2*: If $|\frac{1}{3}-\tau| = \mathrm{O}(\beta)$ as $\beta \to 0$, i.e., $(\frac{1}{3}-\tau) = \pm K_1\beta$ as $\beta \to 0$ (K_1 fixed), then a balance between non-linearity and dispersion requires $\alpha = \mathrm{O}(\beta^2)$ as $\beta \to 0$, i.e., $(\alpha/\beta^2) \to K_2 > 0$ as $\beta \to 0$. It is worth recalling that this same relation has been mentioned in (2.6) which was derived from the dispersion relation (2.4) or (2.5). Then we have the Boussinesq equation (3.39) correct up to $\mathrm{O}(\alpha) = \mathrm{O}(\beta^2)$ as

$$\eta_{tt} - \eta_{xx} - \alpha\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t\,\mathrm{d}x\right)^2\right]_{xx} \mp \frac{K_1}{K_2}\alpha\eta_{xxxx} - \frac{\alpha}{45K_2}\eta_{xxxxxx} = 0. \tag{4.2}$$

This equation is appropriate for $\tau \uparrow \frac{1}{3}$ (i.e., Bond number less than but very close to $\frac{1}{3}$) when the fourth-order dispersive term is negative, and for $\tau \downarrow \frac{1}{3}$ (i.e., Bond number greater than but very close to $\frac{1}{3}$) when the fourth-order dispersive term is positive. The sixth-order Boussinesq equation (4.2) is, perhaps, also non-Hamiltonian and non-conservative. Further, this model is valid on a time-scale of the order $\mathrm{O}(1/\sqrt{\alpha}) = \mathrm{O}(1/\beta)$. Therefore, the time-interval of validity of the model (4.2) is same as that of the model (4.1), as β is same for both the models. However, amplitude of the traveling-wave solutions of the model (4.2) will be smaller than that of the model (4.1). This is because the amplitude parameter α in the model (4.2) is of $\mathrm{O}(\beta^2)$, whereas the amplitude parameter α in the model (4.1) is of $\mathrm{O}(\beta)$, where β is fixed.

4.1. Canonical forms of the Boussinesq equations

At first sight, the fourth-order and sixth-order Boussinesq equations (4.1) and (4.2) look rather complicated due to the presence of non-local operator. But, if we use the following co-ordinate transformation (see Johnson [3])

$$\left.\begin{aligned} X &= \sqrt{\frac{K_2}{K_1}}\left(x + \alpha \int_{-\infty}^{x} \eta(x,t)\,\mathrm{d}x\right) \\ T &= \sqrt{\frac{K_2}{K_1}}t \end{aligned}\right\}, \tag{4.3}$$

and substitute (by arbitrary choice)

$$N = \frac{3}{2}\left(\eta - \alpha\eta^2\right), \tag{4.4}$$

then the fourth-order and sixth-order Boussinesq equations (4.1) and (4.2) are transformed to the following canonical forms

$$N_{TT} - N_{XX} - \alpha\left(N^2\right)_{XX} \mp \alpha N_{XXXX} = 0, \tag{4.5}$$

and

$$N_{TT} - N_{XX} - \alpha\left(N^2\right)_{XX} \mp \alpha N_{XXXX} - \epsilon^2\alpha N_{XXXXXX} = 0, \tag{4.6}$$

where we have again neglected terms of $\mathrm{O}(\alpha^2)$ and higher. Here $\epsilon^2 = K_2/45K_1^2$. It may seem at first that different physical meaning of N, X, T makes a direct comparison with original equations a non-trivial task. However, it is not so. Some explanation of these new coordinates, in particular X and N, will be helpful at this point. It follows from spatial scaling in (4.3) that $\partial X/\partial x = 1 + \alpha\eta$ (up to a constant) which is the free surface elevation after scaling introduced in Section 2. Therefore, the new independent variable X, defined in (4.3), as a function of x at any fixed T is a measure of the net mass of fluid to the left of x (up to a constant multiple) which has one-to-one correspondence with x. The dependent variable N as defined in (4.4) can be viewed as a 'pseudo-surface elevation'. The way N is defined, it is always positive since $\alpha \ll 1$ and $\eta \ll 1$. Eq. (4.4) is quadratic in η and has two real roots for small values of N (N should be even smaller than η in general): one root is closer to one ($\eta \approx 1$) and the other root is very small ($\eta \ll 1$). It is the second root, $\eta \ll 1$, that is physically consistent and should be chosen in going from new coordinate N to η. Note that if one can solve the transformed equations for $N(X,T)$, it can then be inverted, as mentioned above, to find $\eta(X,T)$. Then the corresponding value of x can be found by appropriately integrating $\partial X/\partial x = 1 + \alpha\eta$ since η is known as a function of X.

The fourth-order and sixth-order Boussinesq equations (4.5) and (4.6) can be written even in more canonical forms using scalings $X \to \alpha^{1/2}X$, $T \to \alpha^{1/2}T$, and $N \to \alpha^{-1}N$. Under this rescaling, form of Eq. (4.5) scales into itself with $\alpha = 1$. These fourth-order Boussinesq equations are known to be completely integrable and possess solitonic solutions though solitonic interactions for these equations and, for that matter, initial value problem for these equations are not always non-trivial (see [9]). The constants of motion for these equations involving the new variable N naturally can be written in terms of the original variable η, if necessary. These can be useful for numerical purposes as well in monitoring accuracy of numerical solutions while directly simulating these fourth-order Boussinesq equations. However, as mentioned before none of these constants may correspond to energy-conserving invariant for Eq. (3.39).

It is worth summarizing here that both Eqs. (4.5) and (4.6) represent the bi-directional propagation of small amplitude long capillary-gravity waves on the surface of shallow water. Moreover, Eq. (4.5) is appropriate with the negative sign in the last term when $0 \leqslant \tau \ll 1/3$ and with the positive sign in the last term when $1/3 \ll \tau$, respectively. Similarly, Eq. (4.6) is appropriate with the negative sign in the fourth-order term when $\tau \uparrow 1/3$ and with the positive sign in the fourth-order term when $\tau \downarrow 1/3$. Usually, the coefficient of the sixth-order term in Eq. (4.6) is very small and hence this equation can be considered as a singular perturbation of Eq. (4.5). It is also closely related to the singularly perturbed fifth-order KdV equation (see Hunter and Scheurle [8]) which supports only uni-directional waves (see the subsection below). The sixth-order Boussinesq equation (4.6) with negative fourth-order dispersive term was originally introduced on a heuristic ground by Daripa and Hua [9] as a dispersive regularization of the ill-posed fourth-order Boussinesq equation (4.5) (with negative fourth-order dispersive term).

4.2. *Conversion of Boussinesq equations into KdV equations*

The above Boussinesq equations (4.5) and (4.6) can be converted into corresponding KdV equations using the far-field co-ordinate transformations

$$\xi = X - T \quad \text{and} \quad \tau = \alpha T. \tag{4.7}$$

The transformation (4.7) describes a wave which changes slowly in a reference frame moving with velocity one (the non-dimensional shallow water velocity). The leading order terms in the transformed equations correspond to the following KdV equations.

$$N_\tau + NN_\xi + \frac{1}{2}N_{\xi\xi\xi} = 0, \tag{4.8}$$

and

$$N_\tau + NN_\xi + \frac{1}{2}N_{\xi\xi\xi} + \frac{K_2}{90K_1^2}N_{\xi\xi\xi\xi\xi} = 0. \tag{4.9}$$

If we further use the change of variables

$$\xi \to \frac{\xi}{\sqrt{2\delta}}, \qquad \tau \to \frac{\tau}{\delta\sqrt{2\delta}}, \qquad N \to \delta N, \tag{4.10}$$

where δ is an arbitrary scaling parameter, then the KdV equations (4.8) and (4.9) reduce to the following forms.

$$N_\tau + NN_\xi + N_{\xi\xi\xi} = 0, \tag{4.11}$$

and

$$N_\tau + NN_\xi + N_{\xi\xi\xi} + \frac{2K_2\delta}{45K_1^2}N_{\xi\xi\xi\xi\xi} = 0. \tag{4.12}$$

Eqs. (4.9) and (4.12) are two versions of the fifth-order KDV equation originally derived by Hunter and Scheurle [8]. Eq. (4.12) is exactly same as Eq. (2.29) in Hunter and Scheurle [8] if we use the notation $\epsilon^2 = 2K_2\delta/(45K_1^2)$. The other generalized higher-order KdV equations proposed by Kichenassamy [16] can be obtained from the generalized higher-order Boussinesq equation by restricting the waves to a submanifold traveling only in one direction, as in Olver [13,14].

5. On the solutions of the model equations

As discussed in the previous section, the α in Eqs. (4.5) and (4.6) can be set to 1 by simple rescaling. We rewrite these equations with $\alpha = 1$ using notations f, t and x for N, T and X respectively.

$$f_{tt} - f_{xx} - (f^2)_{xx} - f_{xxxx} = 0, \tag{5.13}$$

and

$$f_{tt} - f_{xx} - (f^2)_{xx} - f_{xxxx} - \epsilon^2 f_{xxxxxx} = 0. \tag{5.14}$$

In the above we have retained the equations with only one sign (-ve) preceding the fourth-order derivative term since these are most relevant ones (the ones with + sign for this derivative term can be similarly analyzed). The fourth-order equation (5.13) is a bad Boussinesq equation because of the short-wave instability (growth rate proportional to k^2 as $k \to \infty$) which can easily be seen by writing down its linearized dispersion relation. Except for arbitrary constant values, known exact solutions are of soliton-type (singlet as well as doublet) which can be found in Hirota [24] and Chu et al. [25]. A family of solitary wave solutions of the bad Boussinesq equation (5.13) is given by (see Hirota [24] and Chu et al. [25])

$$f^s(x,t) = 6\gamma^2 \operatorname{sech}^2\{\gamma(x - ct + x_0)\} + \left(b - \frac{1}{2}\right), \tag{5.15}$$

where $6\gamma^2$ is the amplitude of the solitary wave, b is a free parameter and $c = \mp\sqrt{2(b + 2\gamma^2)}$ is the speed of the solitary wave. Solitary wave corresponding to $b = 1/2$ and $x_0 = 0$ is given by

$$f^s(x,t) = 6\gamma^2 \operatorname{sech}^2\{\gamma(x - ct)\}. \tag{5.16}$$

The speed of this solitary wave is given by $c = \mp\sqrt{1 + 4\gamma^2}$. The initial value problem associated with the bad Boussinesq equation in general is not well-posed for an arbitrary initial data. Choosing initial data for which it is well-posed is difficult except for data corresponding to exact known solutions such as (5.16). Even in these cases when the problem is well-posed, the Fourier amplitudes of round-off error during numerical computation can get multiplied by e^{tk^2} for large k and thus can lead to instabilities. Suppression of these catastrophic instabilities by use of smart filters has been discussed in considerable detail in Daripa and Hua [9].

The sixth-order equation (5.14) differs from the fourth-order one in the singular perturbation term. It is not difficult to see from its dispersion relation that the perturbation term removes the short-wave instability. However, effect of the perturbation term on the classical solitary-type solutions of the unperturbed equation (5.13) is to qualitatively change the behavior of these solutions in the far-field: exponentially small amplitude oscillations in the far-field develop in the solutions which would otherwise be absent in the case of unperturbed equation (5.13). These are called non-local solitary waves. To see this, substitute $f(x,t) = f(x-ct)$ in (5.14) and then integrate the resulting ODE twice resulting in the singular perturbation equation for the ODE (we have taken the constants of integration zero without any loss of generality).

$$\left(1-c^2\right)f + f^2 + f_{xx} + \epsilon^2 f_{xxxx} = 0, \tag{5.17}$$

where with slight abuse of notation we have used x for $(x-ct)$. Using regular perturbation series it has been shown in Dash and Daripa [7] that qualitative shape of the core (hump) of the solitary wave remains intact except for $O(\epsilon)$ quantitative effect. In the far-field, Eq. (5.17) can be linearized assuming that f is small. Then we get after using $c^2 - 1 = 4\gamma^2$

$$-4\gamma^2 f + f_{xx} + \epsilon^2\, f_{xxxx} = 0 \quad \text{as } x \to \pm\infty, \tag{5.18}$$

which has solutions of the form $f = \exp(ipx)$ where p is obtained from the roots of the polynomial equation: $\epsilon^2 p^4 - p^2 = 4\gamma^2$. For $\epsilon = 0$ case, two roots are purely imaginary which correspond to decaying and growing behavior of f at infinity. Only the decaying one is meaningful for the classical solitary wave case. For $\epsilon \neq 0$, it has two real roots p_r (which correspond to the oscillatory behavior of f at infinity) and two purely imaginary roots p_i (which correspond to decaying and growing behavior of f at infinity) which are given by

$$p_r^2 \approx \frac{1}{\epsilon^2} + 4\gamma^2, \qquad p_i^2 \approx -4\gamma^2. \tag{5.19}$$

Therefore, for f to be bounded, it must be of the form

$$f = A_{1\pm}\cos\left(\frac{q}{\epsilon}x\right) + A_{2\pm}\sin\left(\frac{q}{\epsilon}x\right) \quad \text{as } x \to \pm\infty, \tag{5.20}$$

where $A_{1\pm}$ and $A_{2\pm}$ are some ϵ-dependent unknown constants and $q = |p_r|\epsilon$. From $(5.19)_1$, it follows that $q \to 1$ as $\epsilon \to 0$. So, the frequency of oscillations $|p_r| = \frac{q}{\epsilon} \to \frac{1}{\epsilon}$ as $\epsilon \to 0$, and hence, the far-field oscillations are very fast. It is clear from Eq. (5.20) that, in general, there will be oscillatory behaviors on both sides at infinity. Also, the amplitude of oscillations on either ends may be different. More details on these local and non-local solitary wave solutions of these equations can be found in [9] and [7]. Solutions of initial value problems associated with these equations are topics of future research and fall outside the scope of this paper.

6. Summary and discussion

Surface tension, in general, is supposed to damp short waves on physical ground. So, if a model based on zero surface tension is illposed due to catastrophic growth rate of short waves, it is natural to include surface tension effect in the model in the hope that it will damp short waves, but it may not always do so, in particular for all values of surface tension, due to other approximations that lead to these new models. These models may not even satisfy many of the desired properties of the original model as discussed in the next section. This cannot be any further from truth, in particular, in models that are derived by series expansion and truncation at various orders. In fact, models discussed in this paper are derived in this way. For example, consider the classical fourth-order Boussinesq equation (i.e., our Eq. (3.39) correct up to $O(\alpha, \beta)$ and with $\tau = 0$). This equation is severely ill-posed. However, if we include the surface tension effect, the resulting 4th-order Boussinesq equation (i.e., our Eq. (3.39) correct up to $O(\alpha, \beta)$) is well-posed for $\tau > 1/3$, though it still remains ill-posed for $\tau < 1/3$. The former case (the $\tau > 1/3$ case) may not be physically relevant, as for shallow water waves the surface tension effect is small. Thus, inclusion of surface tension effect does make the equation well-posed but only when the bond number $\tau > 1/3$ which is not physically relevant as mentioned above. In order for surface tension, however small, to be able to damp short waves in the model equation, it is necessary to extend the equation beyond the fourth-order. This motivated us for a systematic extension of the fourth-order equation to a sixth-order model (3.39) which is physically relevant for small values of surface tension. This Eq. (3.39), correct up to $O(\alpha^2, \alpha\beta, \beta^2)$ is well-posed for $\tau < 2/5$, and thus for these small values of surface tension which is the case for shallow water waves.

If the linear dispersion relation is expanded in powers of k^2 as we have very clearly laid out in Eq. (2.5) (see also Eq. (2.4)), then one can see that illposedness arises if truncated at orders k^4h^2 but wellposedness restored at next order k^6h^4. One can see from Eq. (2.5) that the fourth-order ($O(k^4)$) term is negative for $\tau < 1/3$, where as the sixth-order term is positive for $\tau < 2/5$. This suggests that if we truncate Eq. (2.5) to order of k^4h^2, then the resulting equation will be ill-posed for $\tau < 1/3$ and well-posed for $\tau > 1/3$. However, if we truncate Eq. (2.5) to order of k^6h^4, then the resulting equation will be well-posed for $\tau < 2/5$ and ill-posed for $\tau > 2/5$. Thus the extended equation serves as a better model equation for shallow and long water wave problems where surface tension effects are small. Again, we emphasize that the fourth-order Boussinesq equation for non-zero surface tension (as mentioned before) is ill-posed for small values of surface tension.

By keeping the balance $\alpha = \beta$ explicitly, we will still get our main extended equation (3.39) *with $\alpha = \beta$ there*. In that case, we will have three variables η_0, η_1, and η_2 in the series rather than six: η_{00}, η_{01}, η_{10}, η_{11}, η_{02}, and η_{20} that appear in this paper. This certainly simplifies the perturbation series analysis to a great extent. However, our goal is not just to derive any sixth-order equation. Our goal is also to see under what conditions we could derive the sixth-order singularly perturbed Boussinesq equation (4.2) and its canonical form (4.6), originally introduced and extensively studied by Daripa and Hua (see Ref. [9]) as a heuristic regularization of the fourth-order ill-posed Boussinesq equation. As we have shown in Section 4.2, unless we have the balance $\alpha = \beta^2$, (i.e., the balance of non-linearity and dispersion) we will not get the sixth-order Boussinesq equation introduced in Ref. [9]. These sixth-order Boussinesq equations (4.2) and (4.6) are also much simpler than the sixth-order equation arising from Eq. (3.39) by directly setting $\alpha = \beta$ there. Note that sixth-order Boussinesq equation (4.2) does not include the non-linear terms $O(\alpha^2)$ and $O(\alpha\beta)$ of Eq. (3.39) and thus makes the sixth-order water wave models (4.2) and (4.6) much simpler to analyze and study computationally (see [9,2]).

7. Concluding remarks

In this paper, the singularly perturbed sixth-order Boussinesq equation, recently introduced by Daripa and Hua [9], is derived rigorously from the two-dimensional Euler's equations of motion for shallow water waves. In the process, we have provided a general higher-order Boussinesq equation (3.39) which is valid for arbitrary small values of two parameters characterizing small amplitude and long wavelength of the waves separately. This also includes the effect of surface tension and is well posed for small values of surface tension for which $\tau < 2/5$. The general higher-order Boussinesq equation (3.39) is likely to provide accurate description of the dispersive non-linear waves for a wider range of parameters than otherwise possible with classical Boussinesq type equations. The range of applicability of this general equation is yet to be explored and falls outside the scope of this paper.

The reduction of this general higher-order Boussinesq equation (3.39) into the sixth-order Boussinesq equation in appropriate limits has been addressed in this paper. It is shown that the sixth-order Boussinesq equation describes the bi-directional propagation of small amplitude long capillary-gravity waves for Bond number τ very close to 1/3. This equation with negative fourth-order dispersive term provides the dispersive regularization of the ill-posed classical fourth-order Boussinesq equation discussed in Daripa and Hua [9]. The traveling wave solutions of this equation do not vanish in the far-field. Such solutions possess small amplitude fast oscillations at distances far from the core of the waves extending up to infinity, well-known as weakly non-local solitary waves (see Dash and Daripa [7]).

In general, dynamical equations arising in non-dissipative media admit Hamiltonian formulation with energy-conserving functional. When standard perturbation methods involving power series in one or more small parameters are applied to develop simpler model equations at various orders, there is no reason to expect that such model equations also admit Hamiltonian formulation and the ones which do are few and more of an exception than the rule. It is possible, but not probable in all cases, that a non-Hamiltonian model equation valid at a specific order can be modified using lower order model equations to a Hamiltonian model equation without changing the specific order of validity of the equation. However, there is no known procedure for doing this in a systematic manner.

Most often, Hamiltonian models are preferred because such models usually possess energy conserving functional which is desired from the viewpoint of physics of such non-dissipative media. In addition, such functionals are useful during numerical simulation of such equations in monitoring the accuracy of numerical solutions in time and/or assessing the appropriateness of numerical schemes used for such simulation purposes. Moreover, many such Hamiltonian models are completely integrable and some even admit soliton-like solutions such as the KdV equation, even though the KdV equation can be derived using standard perturbation methods on the non-dissipative equations of water waves. In fact, such "soliton" equations arise in a wide variety of applied fields as model equations (see [26]). These various reasons have given impetus for the development of methods that can generate Hamiltonian model equations. There have been isolated efforts on a case by case basis in developing such Hamiltonian models, but there is no known general method that can be applied to dynamical equations of non-dissipative media except for the general Hamiltonian perturbation theory which Olver [13] has applied for determining Hamiltonian model equations from non-canonical perturbation expansion of Hamiltonian systems in the context of Boussinesq expansion for long, small amplitude wave in shallow water. Investigation into the Hamiltonian structure of the model equations derived here and development of such or equivalent models using a combination Olver's approach and perturbation methods in double series on the Euler's equations of motion for water waves are worth pursuing which is a topic of future research.

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