A Domain Embedding/Boundary Control Method to Solve Elliptic Problems in Arbitrary Domains

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Abstract

This paper briefly describes some essential theory of a numerical method based on domain embedding and boundary control to solve elliptic problems in complicated domains. A detailed account of this method along with many proofs and numerical results can be found in Badea and Daripa [1].

1 Introduction

In domain embedding methods, irregular domains ω where solutions of problems are sought are embedded into regular domains Ω so that solutions in these embedded domains can be constructed more efficiently, possibly using some fast and accurate solvers on the embedding domains. There are several ways to exploit domain embedding ideas in actual construction of solutions in the irregular domain. For example, domain embedding idea is usually used in conjunction with the optimal boundary or distributed control methods, or the Lagrange multiplier techniques. The use of these embedding methods is now commonplace for solving complicated problems arising in science and engineering. In this regard, it is worth mentioning works of Borgers [2] for Stokes equations, Dinh et. al. [3] for fluid dynamics and electromagnetics, Neittaanmäki et. al. [6] for free boundary problems and optimal design, Young et. al. [7] for the transonic flow calculation, just to mention a few. Due to space limitation, we have not been able to provide here more references on these embedding methods. More references on these methods can be found in Badea and Daripa [1].

Below we first state (section 2) some results related to the formulation of a Dirichlet problem in terms of an optimal boundary control problem. In this formulation, the solution on the auxiliary domain Ω is sought such that it satisfies the boundary conditions on the domain ω. As proved in Badea and Daripa [1] and stated below, this optimal boundary control problem has an unique solution if the controls are taken in a finite dimensional subspace of the space of the boundary conditions on the auxiliary domain. We also state the main theorem related to the proof that the solutions of Dirichlet (or Neumann) problems can be approximated within any prescribed error, however small, by solutions of Dirichlet (or Neumann) problems in the auxiliary domain taking an appropriate subspace for such an optimal control problem.

In Badea and Daripa [1], we have proved that the results obtained for the interior problems hold for the exterior problems. There, we have given some numerical examples for both the interior and the exterior D irichlet problems.

2 Controllability

We briefly outline some key results here for the controlability using the Dirichlet problems as defined below. The proofs of the theorems mentioned here as well as a similar development for the Neumann problems are given in Badea and Daripa [1].

Below we consider a domain embedding and optimal boundary control approach to solve the following elliptic equation:

$$Ay = f \quad \text{in } \omega \tag{1}$$

subject to Dirichlet boundary conditions

$$y = g_\gamma \quad \text{on } \gamma, \tag{2}$$

or Neumann boundary conditions

$$\frac{\partial y}{\partial n_A(\omega)} = h_\gamma \quad \text{on } \gamma, \tag{3}$$

where A is a suitable elliptic operator (see [1]), and $\frac{\partial}{\partial n_A(\omega)}$ is the outward conormal derivative associated with A. Here and below ω, $\Omega \in \mathcal{N}^{(1),1}$ (i.e. the maps defining the boundaries of the domains and their derivatives are Lipschitz continuous) are two bounded domains in $\mathbf{R}^N$ such that $\bar{\omega} \subset \Omega$. Their boundaries are denoted by γ and Γ, respectively. Appropriate function spaces of the solutions are given in Badea and Daripa [1].

The Dirichlet problem (1)–(2) has an unique solution

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which depends continuously on the data

$$|y|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |g_\gamma|_{L^2(\gamma)}\}. \tag{4}$$

Also, under some classical conditions, the Neumann problem (1), (3) has a unique solution which depends continuously on the data

$$|y|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |h_\gamma|_{H^{-1}(\gamma)}\}. \tag{5}$$

We study the controllability of the solution of the above problems in ω with the solution of a Dirichlet problem in Ω. Let

$$\mathcal{U} = L^2(\Gamma) \tag{6}$$

be the space of controls. The state of the system for a control $v \in L^2(\Gamma)$ is given by the solution $y(v) \in H^{1/2}(\Omega)$ of the following Dirichlet problem

$$\begin{array}{lll} Ay(v) = f & \text{in} & \Omega \\ y(v) = v & \text{on} & \Gamma. \end{array} \tag{7}$$

In the case of the Dirichlet problem (1)–(2), the space of observations is taken to be

$$\mathcal{H} = L^2(\gamma), \tag{8}$$

and the cost function is given by

$$J(v) = \frac{1}{2}|y(v) - g_\gamma|^2_{L^2(\gamma)}, \tag{9}$$

where $v \in L^2(\Gamma)$ and $y(v)$ is the solution of problem (7). For the Neumann problem given by (1) and (3), the space of observations is taken to be

$$\mathcal{H} = H^{-1}(\gamma), \tag{10}$$

and the cost function is given by

$$J(v) = \frac{1}{2}\left|\frac{\partial y(v)}{\partial n_A(\omega)} - h_\gamma\right|^2_{H^{-1}(\gamma)}. \tag{11}$$

Proposition 2.1 *A control $u \in L^2(\Gamma)$ satisfies $J(u) = 0$, where the control function is given by (9), if and only if the solution of (7) for $v = u$, $y(u) \in H^{1/2}(\Omega)$ satisfies*

$$\begin{array}{ll} Ay(u) = f & \text{in } \Omega - \bar{\omega}, \\ y(u) = y & \text{on } \gamma, \\ \frac{\partial y(u)}{\partial n_A(\Omega-\bar{\omega})} + \frac{\partial y}{\partial n_A(\omega)} = 0 & \text{on } \gamma, \end{array} \tag{12}$$

and

$$y(u) = y \quad \text{in } \omega, \tag{13}$$

where y is the solution of the Dirichlet problem defined by (1) and (2) in the domain ω. The same result holds if the control function is given by (11) and y is the solution of the Neumann problem (1) and (3).

Since (12) is not a properly posed problem, it follows from the above proposition that the optimal control might not exist. However, J. L. Lions proves in [4] (Chap. 2, §5.3, Theorem 5.1) a controllability theorem which can be directly applied to problem (7). Using this controllability theorem, the following theorem (see Badea and Daripa [1]) can be proved, and it justifies controllability of the solutions of problems in ω by the solutions of Dirichlet problems in Ω.

Theorem 2.1 *The set $\{y(v)_{|\omega} : v \in L^2(\Gamma)\}$ is dense, using the norm of $H^{1/2}(\omega)$, in $\{y \in H^{1/2}(\omega) : Ay = f \text{ in } \omega\}$, where $y(v) \in H^{1/2}(\Omega)$ is the solution of the Dirichlet problem (7) for a given $v \in L^2(\Gamma)$.*

In Badea and Daripa [1], the controllability of the solutions of Dirichlet and Neumann problems in ω by Neumann problems in Ω is also discussed, and a theorem similar to Theorem 2.1 in this case is stated and proved there.

3 Controllability with finite dimensional spaces

Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional subspaces of the space $L^2(\Gamma)$ such that given (6) as a space of controls with the Dirichlet problems, we have

$$\bigcup_\lambda U_\lambda \text{ is dense in } \mathcal{U} = L^2(\Gamma). \tag{14}$$

For a $v \in L^2(\Gamma)$ we consider the solution $y'(v) \in H^{1/2}(\Omega)$ of the problem

$$\begin{array}{ll} Ay'(v) = 0 & \text{in } \Omega \\ y'(v) = v & \text{on } \Gamma. \end{array} \tag{15}$$

We fix an U_λ. The cost functions J defined by (9) and (11) are differentiable and convex. Consequently, an optimal control

$$u_\lambda \in U_\lambda : J(u_\lambda) = \inf_{v \in U_\lambda} J(v), \tag{16}$$

exists if and only if it is a solution of the equation

$$\begin{array}{l} u_\lambda \in U_\lambda : (y(u_\lambda), y'(v))_{L^2(\gamma)} = \\ (g_\gamma, y'(v))_{L^2(\gamma)} \text{ for any } v \in U_\lambda. \end{array} \tag{17}$$

when the control function is (9), and

$$\begin{array}{l} u_\lambda \in U_\lambda : \left(\frac{\partial y(u_\lambda)}{\partial n_A(\omega)}, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} = \\ \left(h_\gamma, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} \text{ for any } v \in U_\lambda, \end{array} \tag{18}$$

when the control function is (11). Above, $y(u_\lambda)$ is the solution of problem (7) corresponding to u_λ, and $y'(v)$

is the solution of problem (15) corresponding to v. If $y_f \in H^2(\Omega)$ is the solution of the problem

$$\begin{aligned} Ay_f &= f \quad \text{in } \Omega, \\ y_f &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{19}$$

then, for a $v \in L^2(\Gamma)$, we have

$$y(v) = y'(v) + y_f, \tag{20}$$

where $y(v)$ and $y'(v)$ are the solutions of problems (7) and (15), respectively. Therefore, we can rewrite problems (17) and (18) as

$$\begin{aligned} &u_\lambda \in U_\lambda : (y'(u_\lambda), y'(v))_{L^2(\gamma)} = \\ &(g_\gamma - y_f, y'(v))_{L^2(\gamma)}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} &u_\lambda \in U_\lambda : \left(\tfrac{\partial y'(u_\lambda)}{\partial n_A(\omega)}, \tfrac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} = \\ &\left(h_\gamma - \tfrac{\partial y_f}{\partial n_A(\omega)}, \tfrac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)}, \end{aligned} \tag{22}$$

for any $v \in U_\lambda$, respectively.

Lemma 3.1 *For a fixed λ, let $\varphi_1, \ldots, \varphi_{n_\lambda}$, $n_\lambda \in \mathbf{N}$, be a basis of U_λ, and let $y'(\varphi_i)$ be the solution of problem* (15) *for $v = \varphi_i$, $i = 1, \ldots, n_\lambda$. Then $\{y'(\varphi_1)_{|\gamma}, \ldots, y'(\varphi_{n_\lambda})_{|\gamma}\}$ and $\{\frac{\partial y'(\varphi_1)}{\partial n_A(\omega)}|\gamma, \ldots, \frac{\partial y'(\varphi_{n_\lambda})}{\partial n_A(\omega)}|\gamma\}$ are linearly independent sets.*

The following proposition proves the existence and uniqueness of the optimal control when the states of the system are the solutions of the Dirichlet problems.

Proposition 3.1 *Let us consider a fixed U_λ. Then problems* (21) *and* (22) *have unique solutions. Consequently, if the boundary conditions of Dirichlet problems* (7) *lie in the finite dimensional space U_λ, then there exists a unique optimal control of problem* (16) *corresponding to either the Dirichlet problem* (1), (2) *or the Neumann problem* (1), (3).

The following theorem proves the controllability of the solutions of the Dirichlet problems in ω by the solutions of the Dirichlet problems in Ω. In fact, it proves the convergence of the embedding method associated with the optimal boundary control.

Theorem 3.1 *Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional spaces satisfying* (14). *We associate the solution y of the Dirichlet problem* (1), (2) *in ω with problem* (16), *in which the cost function is given by* (9). *Also, the solution y of the Neumann problem* (1), (3) *is associated with problem* (16), *in which the cost function is given by* (11). *In both cases, there exists a positive constant C, and for any given $\varepsilon > 0$ there exists U_{λ_ε} such that*

$$|y(u_{\lambda_\varepsilon})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon,$$

where $u_{\lambda_\varepsilon} \in U_{\lambda_\varepsilon}$ is the optimal control of the corresponding problem (16) *with $\lambda = \lambda_\varepsilon$, and $y(u_{\lambda_\varepsilon})$ is the solution of problem* (7) *with $v = u_{\lambda_\varepsilon}$.*

Using the basis $\varphi_1, \cdots, \varphi_{n_\lambda}$ of the space U_λ, we define the matrix

$$\Pi_\lambda = ((y'(\varphi_i), y'(\varphi_j))_{L^2(\gamma)})_{1 \le i,j \le n_\lambda} \tag{23}$$

and the vector

$$l_\lambda = ((g_\gamma - y_f, y'(\varphi_i))_{L^2(\gamma)})_{1 \le i \le n_\lambda}. \tag{24}$$

Then problem (21) can be written as

$$\xi_\lambda = (\xi_{\lambda,1}, \cdots, \xi_{\lambda,n_\lambda}) \in \mathbf{R}^{n_\lambda} : \Pi_\lambda \xi_\lambda = l_\lambda. \tag{25}$$

Consequently, using Theorem 3.1, the solution y of problem (1), (2) can be obtained within any prescribed error by setting the restriction to ω of

$$y(u_\lambda) = \xi_{\lambda,1} y'(\varphi_1) + \cdots + \xi_{\lambda,n_\lambda} y'(\varphi_{n_\lambda}) + y_f, \tag{26}$$

where $\xi_\lambda = (\xi_{\lambda,1}, \cdots, \xi_{\lambda,n_\lambda})$ is the solution of algebraic system (25). Above, y_f is the solution of problem (19) and $y'(\varphi_i)$ are the solutions of problems (15) with $v = \varphi_i$, $i = 1, \cdots, n_\lambda$.

An algebraic system (25) is also obtained in the case of problem (22). This time the matrix of the system is given by

$$\Pi_\lambda = \left(\left(\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)}, \frac{\partial y'(\varphi_j)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)}\right)_{1 \le i,j \le n_\lambda}, \tag{27}$$

and the free term is

$$\begin{aligned} &l_\lambda = \\ &\left(\left(h_\gamma - \tfrac{\partial y_f}{\partial n_A(\omega)}, \tfrac{\partial y'(\varphi_i)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)}\right)_{1 \le i \le n_\lambda}. \end{aligned} \tag{28}$$

Therefore, using Theorem 3.1, the solution y of problem (1), (3) can be estimated by (26). Also, y_f is the solution of problem (19), and $y'(\varphi_i)$ are the solutions of problems (15) with $v = \varphi_i$, $i = 1, \ldots, n_\lambda$.

Remark 3.1 We have defined y_f as a solution of problem (19) in order to have $y(v) = y'(v) + y_f$ or $\frac{\partial y(v)}{\partial n_A(\Omega)} = \frac{\partial y'(v)}{\partial n_A(\Omega)} + \frac{\partial y_f}{\partial n_A(\Omega)}$, respectively, on the boundary Γ. In fact, we can replace $y(v)$ by $y'(v) + y_f$ in the cost functions (9) and (11) with $y_f \in H^2(\Omega)$ satisfying only

$$Ay_f = f \text{ in } \Omega, \tag{29}$$

and the results obtained in this section still hold.

4 Approximate observations in finite dimensional spaces

In solving problems (21), (22), we require an appropriate interpolation which makes use of the values of $y'(v)$ computed only at some points on the boundary γ. We show below that using these interpolations, i.e., observations in finite dimensional subspaces, we can obtain the approximate solutions of problems (1), (2) and (1), (3).

As in the previous sections, we deal with the case when the states of the system is given by the Dirichlet problem (7). Let U_λ be a fixed finite dimensional subspace of $\mathcal{U} = L^2(\Gamma)$ with the basis $\varphi_1, \cdots, \varphi_{n_\lambda}$.

Let us assume that for the problem (1), (2), we choose a family of finite dimensional spaces $\{H_\mu\}_\mu$ such that

$$\bigcup_\mu H_\mu \text{ is dense in } \mathcal{H} = L^2(\gamma). \tag{30}$$

Similarly, for problem (1), (3) we choose the finite dimensional spaces $\{H_\mu\}_\mu$ such that

$$\bigcup_\mu H_\mu \text{ is dense in } \mathcal{H} = H^{-1}(\gamma). \tag{31}$$

The subspace H_μ given in (30) and (31) is a subspace of $\mathcal{H}$ given in (8) and (10), respectively.

An appropriate choice of H_μ is made based on the problem to be solved as discussed above. For a given φ_i, $i = 1, \cdots, n_\lambda$, we consider below the solution $y'(\varphi_i)$ of problem (15) corresponding to $v = \varphi_i$ and we approximate its trace on γ by $y'_{\mu,i}$.Also, the approximation of $\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)}$ on γ is denoted by $\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}$.

Since the system (25) has a unique solution, the determinants of the matrices Π_λ given in (23) and (27) are nonzero. Consequently, if $|y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}$ or $|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}|_{H^{-1}(\gamma)}$ are small enough, then the matrices

$$\Pi_{\lambda\mu} = ((y'_{\mu,i}, y'_{\mu,j})_{L^2(\gamma)})_{1\le i,j\le n_\lambda} \tag{32}$$

and

$$\Pi_{\lambda\mu} = \left(\left(\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}, \frac{\partial y'_{\mu,j}}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)}\right)_{1\le i,j\le n_\lambda} \tag{33}$$

have nonzero determinants. In this case, each of the algebraic systems

$$\xi_{\lambda\mu} = (\xi_{\lambda\mu,1}, \ldots, \xi_{\lambda\mu,n_\lambda}) \in \mathbf{R}^{n_\lambda} : \Pi_{\lambda\mu}\xi_{\lambda\mu} = l_{\lambda\mu} \tag{34}$$

has a unique solution. In this system, the free term is

$$l_{\lambda\mu} = ((g_{\gamma\mu} - y_{f\mu}, y'_{\mu,i})_{L^2(\gamma)})_{1\le i\le n_\lambda} \tag{35}$$

if the matrix $\Pi_{\lambda\mu}$ is given by (32) and

$$l_{\lambda\mu} = \left(\left(h_{\gamma\mu} - \frac{\partial y_{f\mu}}{\partial n_A(\omega)}, \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)}\right)_{1\le i\le n_\lambda} \tag{36}$$

if the matrix $\Pi_{\lambda\mu}$ is given by (33). Above, we have denoted by $g_{\gamma\mu}$ and $h_{\gamma\mu}$ some approximations in H_μ of g_γ and h_γ, respectively. Also, $y_{f\mu}$ and $\frac{\partial y_{f\mu}}{\partial n_A(\omega)}$ are some approximations of y_f and $\frac{\partial y_f}{\partial n_A(\omega)}$ in the corresponding H_μ of $L^2(\gamma)$ and $H^{-1}(\gamma)$, respectively, with $y_f \in H^2(\Omega)$ satisfying (29).

The solution y of problems (1), (2) and (1), (3) can be approximated with the restriction to ω of

$$y(u_{\lambda\mu}) = \xi_{\lambda\mu,1}y'(\varphi_1) + \cdots + \xi_{\lambda\mu,n_\lambda}y'(\varphi_{n_\lambda}) + y_f, \tag{37}$$

where $\xi_\lambda = (\xi_{\lambda\mu,1}, \ldots, \xi_{\lambda\mu,n_\lambda})$ is the solution of appropriate algebraic system (34).

Using Theorem 3.1, we can prove the following theorem which estimates the error depending on the approximation on the boundary γ of the domain ω of both the boundary conditions and the solutions on Ω.

Theorem 4.1 *Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional spaces satisfying* (14). *Also, we associate problem* (1), (2) *or* (1), (3) *with a family of spaces $\{H_\mu\}_\mu$ satisfying* (30) *or* (31), *respectively. Then, for any $\varepsilon > 0$, there exists λ_ε such that the following hold.*

(i) *If the space H_μ is taken such that $|y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}$, $i = 1, \ldots, n_{\lambda_\varepsilon}$, are small enough, y is the solution of problem* (1)–(2), *and $y(u_{\lambda_\varepsilon\mu})$ is given by* (37), *in which $\xi_{\lambda\mu}$ is the solution of the algebraic system* (34) *with the matrix given by* (32) *and the free term given by* (35), *then the algebraic system* (34) *has a unique solution and*

$$|y(u_{\lambda_\varepsilon\mu})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon + C_{\lambda_\varepsilon}\left(|g_\gamma - g_{\gamma\mu}|_{L^2(\gamma)} + |y_f - y_{f\mu}|_{L^2(\gamma)} + \max_{1\le i\le n_\lambda} |y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}\right).$$

(ii) *If the space H_μ is taken such that $|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}|_{H^{-1}(\gamma)}$, $i = 1, \ldots, n_{\lambda_\varepsilon}$, are small enough, y is the solution of problem* (1)–(3), *and $y(u_{\lambda_\varepsilon\mu})$ is given by* (37) *in which $\xi_{\lambda\mu}$ is the solution of the algebraic system* (34) *with the matrix given by* (33) *and the free term given by* (36), *then the algebraic system* (34) *has a unique solution and*

$$|y(u_{\lambda_\varepsilon\mu})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon + C_{\lambda_\varepsilon}\left(|h_\gamma - h_{\gamma\mu}|_{H^{-1}(\gamma)} + \left|\frac{\partial y_f}{\partial n_A(\omega)} - \frac{\partial y_{f\mu}}{\partial n_A(\omega)}\right|_{H^{-1}(\gamma)} + \max_{1\le i\le n_\lambda}\left|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}\right|_{H^{-1}(\gamma)}\right),$$

where C is a constant and C_{λ_ε} depends on the basis of U_{λ_ε}.

Remark 4.1. Since the matrices $\Pi_{\lambda\mu}$ given by (32) and (33) are assumed to be nonsingular, it follows that $\{y'_{\mu,i}\}_{i=1,\dots,n_\lambda}$ and $\{\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}\}_{i=1,\dots,n_\lambda}$ are some linearly independent sets in $L^2(\gamma)$ and $H^{-1}(\gamma)$, respectively. Consequently, if m_μ is the dimension of the corresponding subspace H_μ, then $n_\lambda \le m_\mu$.

5 Numerical example

The numerical tests refer to the Dirichlet problem

$$\begin{aligned} -\Delta y &= f \text{ in } \omega, \\ y &= g_\gamma \text{ on } \gamma, \end{aligned} \tag{38}$$

where $\omega \subset \mathbf{R}^2$ is a square centered at the origin with sides parallel to the axes and of length of 2 units. The approximate solution of this problem is given by the solution of the Dirichlet problem

$$\begin{aligned} -\Delta y(v) &= f \text{ in } \Omega, \\ y(v) &= v \text{ on } \Gamma, \end{aligned} \tag{39}$$

in which the domain Ω is the disc centered at the origin with radius equal to 2. The solutions of the homogeneous Dirichlet problems in Ω are found by the Poisson formula

$$y(v)(z) = \frac{1}{2\pi r}\int_{|\zeta|=r} v(\zeta)\frac{r^2-|z|^2}{|z-\zeta|^2}dS_\zeta. \tag{40}$$

The circle Γ is discretized with n equidistant points, and $U \subset \mathcal{U} \equiv L^2(\Gamma)$ is taken as the space of the piecewise constant functions. Naturally, an element φ_i in the basis of H is a function defined on Γ which takes the value 1 between the nodes i and $i+1$ and vanishes in the rest of Γ. The square γ is also discretized with m equidistant points, and $H \subset \mathcal{H} \equiv L^2(\gamma)$ is taken as the space of the continuous piecewise linear functions. Evidently, the inclusions in (14) and (30) are dense because the union of the spaces (over some sequence of mesh size approaching zero) of continuous piecewise linear or piecewise constant functions is dense in L^2.

The values of the integrals in the Poisson formula at the points on γ are calculated using the numerical integration with three nodes. The integrals in the inner products in $L^2(\gamma)$ are calculated using an exact formula when H is the space of the continuous piecewise linear functions. In particular, if we have on γ two continuous piecewise linear functions y_1 and y_2 such that

$$\begin{aligned} y_1(x) &= m_1^k(x-x_k)+y_1^k, \\ y_2(x) &= m_2^k(x-x_k)+y_2^k \end{aligned} \tag{41}$$

n	$\mathrm{err_d}$	$\mathrm{err_b}$
80	.36692E-07	.15956E-06
72	.46271E-08	.41101E-07
60	.14682E-09	.25103E-08
45	.12475E-08	.54357E-08
40	.64352E-12	.11638E-07
36	.67121E-12	.11648E-06
30	.12371E-05	.33923E-05
24	.39543E-12	.19851E-04
18	.10609E-03	.43901E-03
12	.29916E-10	.54208E-02
10	.94618E-02	.17096E-01

Table 1: Table 5.1. Relative errors for the Dirichlet problem.

for $x \in [x_k, x_{k+1}]$, $k = 1\dots,m$, then

$$\int_\gamma y_1y_2 = h\sum_{k=1}^m \Big[y_1^ky_2^k + \frac{h^2}{3}m_1^km_2^k + \frac{h}{2}(m_1^ky_2^k+m_2^ky_1^k)\Big], \tag{42}$$

where $h = x_{k+1}-x_k$ is the mesh size on γ.

All computations below have been performed in fifteen digit arithmetics (double precision).

In this example, we choose the exact solution to be $u(x_1,x_2) = x_1^2+x_2^2$. Hence $g_\gamma(x_1,x_2) = x_1^2+x_2^2$, and $f = -4$. We have taken $y_f = 2x_1^2$ as a solution of the inhomogeneous equation in Ω. It has been compared with the computed one at 19 equidistant points on a diagonal of the square: (-1.4,-1.4),. . .,(0,0),. . .,(1.4,1.4). Below $\mathrm{err_d}$ denotes the maximum of the relative errors between the exact and the computed solutions at these 19 considered points in the domain ω. A similar error only on the boundary γ is denoted by $\mathrm{err_b}$.

Table 5.1 shows errors $\mathrm{err_d}$ and $\mathrm{err_b}$ against n, the number of the equidistant points on Γ which is the dimension of the finite dimensional space U. Recall that Γ is boundary of the embedding domain Ω. All these computations use a mesh size of 0.1 on γ. It corresponds to $m = 80$, the number of equidistant points on γ, which is the dimension of the finite dimensional space H. The smallest diagonal element during the Gauss elimination method is of the order 10^{-17} for $n = 80$ and $n = 72$, and of the order 10^{-14} for $n = 60$. It is greater than 10^{-10} for $n = 10,\dots,45$. We should mention that in the cases when $n > 60$, where the last pivot is very small, we notice an increase in error. In all these cases the error $\mathrm{err_b}$, which can be calculated for any example even when the exact solution is not known, is a good indicator of the computational accuracy.

In the above example, the right-hand side f of the equation in ω is given by an exact algebraic formula, and it was extended in Ω by the same formula. Moreover, we have had for this simple example an exact solution y_f of the inhomogeneous equation in Ω, which could be exactly evaluated at the mesh points of the boundary γ of the domain ω. Also, the solutions of the homogeneous problems in Ω, given by the above Poisson formula, could be evaluated directly at these mesh points. In other much more complicated examples in [1] we study the effect of various extensions of f in Ω on the computed solutions in ω. Therefore, in those examples, the solution of the problem in Ω could be computed only at some nodes of a regular mesh over Ω, and their values at the mesh points on γ are calculated by interpolation. We could not give these example here because of the limitations on the numbers of pages (6 pages at most).

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