Relevance of the Sixth-Order Boussinesq Equation for Water Waves

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Abstract

We analyze the two-dimensional Euler's equations of motion governing the capillary-gravity waves on the surface of shallow water. For Bond number less than but very close to 1/3, we show that the bi-directional propagation of small amplitude and long waves is appropriately described by the sixth-order Boussinesq equation which was recently introduced on a heuristic ground by Daripa and Hua [Appl. Math. Comput. **101** (1999), 159–207] as a dispersive regularization of the fourth-order ill-posed (also known as classical) Boussinesq equation.

Keywords: Water Waves, Sixth-Order Boussinesq Equation, Bi-directional Wave Propagation,

1 Introduction

The fourth-order (also known as classical) Boussinesq equation (see Johnson [6] and Whitham [8])

$$\eta_{tt} = \eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx}, \tag{1.1}$$

arises in the context of asymptotic description of bi-directional propagating small amplitude long capillary-gravity water waves. This equation possesses solitary wave solutions. As an initial value problem, it suffers from severe catastrophic short wave instability. The linearized version of equation (1.1) admits solutions of the form $e^{\sigma t + ikx}$ with short wave instability $\sigma \approx k^2$ as $k \to \infty$. One of the consequences of this short wave instability is difficulty in numerically constructing even known solutions, such as the classical solitary wave solutions (see Daripa and Hua [2]), for short time. This is not so surprising since this equation may not have classical solutions even for short time for arbitrary initial data. The exact known solitary wave solutions are probably isolated solutions of this equation. Therefore, in the past, there have been attempts to regularize this equation in ways that are physically more meaningful.

One of the ways to regularize equation (1.1) is to modify this equation by replacing η_{xxxx} by η_{xxtt} (see Whitham [8]). Recently, Daripa and Hua [2] introduced the following sixth-order Boussinesq equation

$$\eta_{tt} = \eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx} + \epsilon^2 \eta_{xxxxxx}, \tag{1.2}$$

as a dispersive regularization of equation (1.1). Here ϵ is a suitably chosen parameter. This regularization (1.2) is based on a heuristic ground and Daripa and Hua [2] does not provide any mathematical derivation as to the origin of this equation. Since an ill-posed equation can be regularized in an ad-hoc fashion in many ways some of which may have no physical relevance to the problem, it is appropriate here to address this issue in relation to the above equation (1.2).

In this paper, we address the physical relevance of this sixth-order Boussinesq equation (1.2) in the context of water waves. In particular, we show in section 2 that equation (1.2) actually describes the bi-directional propagation of small amplitude long capillary-gravity waves on the surface of shallow water for Bond number less than but very close to 1/3. In section 3, we briefly comment on the nature of traveling wave solutions of equation (1.2) based on the recent extensive study of Daripa and Dash [3].

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2 Derivation of Sixth-Order Boussinesq Equation

In this section, we present a formal derivation of the sixth-order Boussinesq equation (1.2) from the two-dimensional Euler's equations of motion for capillary-gravity shallow water waves in the limits of small amplitude and long wavelength with Bond number less than but very close to 1/3.

Let $z = 0$ represent the bottom topography and $z = h(x,t) = h_0 + a\eta(x,t)$ represent the free water surface, where h_0 is the height of the undisturbed water surface, a is the amplitude of the surface wave and $\eta(x,t)$ is the free surface elevation from its undisturbed location. Let (u,w) represent the velocity field in (x,z) coordinate. We use the following non-dimensionalization

$$\left.\begin{array}{l} x \to lx, \quad z \to h_0 z, \quad t \to \frac{l}{\sqrt{gh_0}}t, \quad u \to \frac{a}{h_0}\sqrt{gh_0}u, \\ w \to \left(\frac{a}{h_0}\right)\left(\frac{h_0}{l}\right)\sqrt{gh_0}w, \quad p \to p_a + \rho g(h_0 - z) + \frac{a}{h_0}(\rho g h_0)p, \end{array}\right\} \tag{2.1}$$

where l is the wavelength of the surface wave, g is the acceleration due to gravity, ρ is the density of the fluid, p is the pressure field, and p_a is the atmospheric pressure. In non-dimensional form, the Euler's equations of motion governing the capillary-gravity shallow water waves (see Johnson [6]) are given by

$$\left.\begin{array}{rcl} u_t + \alpha(uu_x + wu_z) &=& -p_x, \\ \beta\left[w_t + \alpha(uw_x + ww_z)\right] &=& -p_z, \\ u_x + w_z &=& 0. \end{array}\right\} \tag{2.2}$$

The corresponding kinematic and dynamic boundary conditions are given by

$$\left.\begin{array}{l} w = 0 \quad \text{at} \quad z = 0, \\ w = \eta_t + \alpha u \eta_x \quad \text{at} \quad z = 1 + \alpha\eta, \\ p = \eta - \beta\tau\frac{\eta_{xx}}{[1+\alpha^2\beta\eta_x^2]^{3/2}} \quad \text{at} \quad z = 1 + \alpha\eta. \end{array}\right\} \tag{2.3}$$

Here $\alpha = a/h_0$ (amplitude parameter), $\beta = (h_0/l)^2$ (wavelength parameter) and $\tau = \Gamma/\rho g h_0^2$ (Bond number) where Γ is the surface tension coefficient.

The linearized version of the above equations admits solutions for η of the form $Ae^{ikx - i\omega t}$ provided the following dispersion relation holds (see Whitham [8])

$$\omega^2 = \frac{c_0^2}{h_0^2}\left[(1 + \tau k^2 h_0^2)\, kh_0 \tanh(kh_0)\right], \tag{2.4}$$

where $c_0 = \sqrt{gh_0}$. In the long wavelength limit (i.e., $kh_0 \ll 1$), we have

$$\omega^2 = c_0^2 k^2 \left[1 - \left(\frac{1}{3} - \tau\right)k^2h_0^2 + \frac{1}{3}\left(\frac{2}{5} - \tau\right)k^4h_0^4 - \frac{2}{15}\left(\frac{17}{42} - \tau\right)k^6h_0^6 + \cdots\right]. \tag{2.5}$$

This indicates that the leading order dispersion term in the equation for η is of order $\left(\frac{1}{3} - \tau\right)k^2h_0^2$, i.e., $\left(\frac{1}{3} - \tau\right)\beta$. Therefore, the leading order dispersion term is $O(\beta)$ if $\left(\frac{1}{3} - \tau\right) = O(1)$ and $O(\beta^2)$ if $\left(\frac{1}{3} - \tau\right) = O(\beta)$ However, the non-linear term is always of order α irrespective of the value of Bond number τ. Therefore, a balance between the non-linear and the dispersive effects (which is necessary to model a solitary wave) requires that $\alpha = O(\beta)$ when $\left(\frac{1}{3} - \tau\right) = O(1)$ and $\alpha = O(\beta^2)$ when $\left(\frac{1}{3} - \tau\right) = O(\beta)$. Thus, for Bond number less than but very close 1/3 (i.e., for $\tau \uparrow 1/3$), we need to have

$$\left(\frac{1}{3} - \tau\right) = K_1\beta \quad \text{and} \quad \alpha = K_2\beta^2 \quad \text{as} \quad \beta \to 0, \tag{2.6}$$

with non-zero constants K_1 and K_2 fixed. Under the conditions (2.6), we can write the governing equations (2.2) in the form

$$\left.\begin{array}{rcl} u_t + K_2\beta^2(uu_x + wu_z) &=& -p_x, \\ \beta\left[w_t + K_2\beta^2(uw_x + ww_z)\right] &=& -p_z, \\ u_x + w_z &=& 0, \end{array}\right\} \tag{2.7}$$

and the boundary conditions (2.3), expressed at $z = 0$ and $z = 1$ using Taylor series expansion, in the form

$$\left.\begin{array}{l} w \;=\; 0 \quad \text{at} \quad z \;=\; 0, \\ w \;+\; K_2\beta^2\,\eta\,w_z \;=\; \eta_t \;+\; K_2\beta^2\,\eta_x\,u \;+\; O(\beta^4) \quad \text{at} \quad z \;=\; 1, \\ p \;+\; K_2\beta^2\,\eta\,p_z \;=\; \eta \;-\; \frac{1}{3}\beta\,\eta_{xx} \;+\; K_1\beta^2\,\eta_{xx} \;+\; O(\beta^3) \quad \text{at} \quad z \;=\; 1. \end{array}\right\} \tag{2.8}$$

Below, we derive the necessary sixth-order Boussinesq equation for η from equations (2.7) and (2.8) through a regular perturbation analysis. In doing so, we express the solution $q = (u, w, p, \eta)$ in the form

$$q = q_0 + \beta q_1 + \beta^2 q_2 + \cdots . \tag{2.9}$$

Upon substituting expansions (2.9) for u, w, p and η into equations (2.7) and boundary conditions (2.8), we obtain the following equations and boundary conditions of various orders as coefficients of $\beta^i,\ i = 0, 1, 2$:

$$O(1): \quad \left\{\begin{array}{l} u_{0t} = -p_{0x} \\ p_{0z} = 0 \\ u_{0x} + w_{0z} = 0, \end{array}\right\}, \quad \left\{\begin{array}{l} w_0 = 0 \ \text{ at } \ z = 0 \\ w_0 = \eta_{0t} \ \text{ at } \ z = 1 \\ p_0 = \eta_0 \ \text{ at } \ z = 1 \end{array}\right\}. \tag{2.10, 2.11}$$

$$O(\beta): \quad \left\{\begin{array}{l} u_{1t} = -p_{1x} \\ w_{0t} = -p_{1z} \\ u_{1x} + w_{1z} = 0 \end{array}\right\}, \quad \left\{\begin{array}{l} w_1 = 0 \ \text{ at } \ z = 0 \\ w_1 = \eta_{1t} \ \text{ at } \ z = 1 \\ p_1 = \eta_1 - \frac{1}{3}\eta_{0x} \ \text{ at } \ z = 1 \end{array}\right\}. \tag{2.12, 2.13}$$

$$O(\beta^2): \quad \left\{\begin{array}{l} u_{2t} + K_2(u_0 u_{0x} \\ \qquad + w_0 u_{0z}) = -p_{2x} \\ w_{1t} = -p_{2z} \\ u_{2x} + w_{2z} = 0 \end{array}\right\}, \quad \left\{\begin{array}{l} w_2 = 0 \ \text{ at } \ z = 0 \\ w_2 + K_2\,\eta_0\,w_{0z} = \eta_{2t} \\ \quad + K_2\,\eta_{0x}\,u_0 \ \text{ at } \ z = 1 \\ p_2 + K_2\,\eta_0\,p_{0z} = \eta_2 - \frac{1}{3}\eta_{1xx} \\ \quad + K_1\,\eta_{0xx} \ \text{ at } \ z = 1 \end{array}\right\}. \tag{2.14, 2.15}$$

Below we derive canonical equations governing η_0, η_1, and η_2 from the above set of equations by eliminating the other variables, namely u, w, and p.

- **Equation for $\boldsymbol{\eta_0(x,t)}$** : It is easy to see from equations (2.10) and (2.11) that

$$p_0 = \eta_0, \quad u_{0t} = -\eta_{0x}, \quad w_0 = -z u_{0x}, \quad u_{0x} = -\eta_{0t}, \tag{2.16}$$

and hence we obtain the equation for η_0 as

$$\eta_{0tt} - \eta_{0xx} = 0. \tag{2.17}$$

Therefore, the solution η_0 will be of the traveling wave form $E_0(x-t) + F_0(x-t)$ for some arbitrary function E_0 and F_0.

- **Equation for $\boldsymbol{\eta_1(x,t)}$** : Substituting the expressions for w_{0t} from equation (2.16) in equation (2.12b) and integrating the resulting equation with the help of condition (2.13c), we obtain

$$p_1 = \big(\eta_1 + \frac{1}{6}\eta_{0xx}\big) - \frac{1}{2}\eta_{0xx}\,z^2. \tag{2.18}$$

Equations (2.12a,c) and (2.18) then together give

$$w_{1zt} = -u_{1xt} = p_{1xx} = \big(\eta_{1xx} + \frac{1}{6}\eta_{0xxxx}\big) - \frac{1}{2}\eta_{0xxxx}\,z^2. \tag{2.19}$$

Integrating equation (2.19) with the help of condition (2.13a) and then using condition (2.13b), we obtain the equation for η_1 as

$$\eta_{1tt} - \eta_{1xx} = 0. \tag{2.20}$$

Therefore, the solution η_1 will be of the traveling wave form $E_1(x-t)+F_1(x-t)$ for some arbitrary function E_1 and F_1.

• **Equation for $\boldsymbol{\eta_2(x,t)}$:** Substituting the expression for w_{1t} from equation (2.19) in equation (2.14b) and integrating the resulting equation with the help of condition (2.15c), we obtain

$$p_2 = \big(\eta_2 + \frac{1}{6}\eta_{1xx} + K_1\eta_{0xx} + \frac{1}{24}\eta_{0xxxx}\big) - \frac{1}{2}\big(\eta_{1xx} + \frac{1}{6}\eta_{0xxxx}\big)z^2 + \frac{1}{24}\eta_{0xxxx}\, z^4. \tag{2.21}$$

Equations (2.14a,c) and (2.21) then can be combined to give

$$\begin{aligned} w_{2zt} = -u_{2xt} = p_{2xx} + K_2(u_0u_{0x})_x = \big[&\eta_{2xx} + \frac{1}{6}\eta_{1xxxx} + K_1\eta_{0xxxx} \\ &+ \frac{1}{24}\eta_{0xxxxxx} + K_2(u_0u_{0x})_x\big] - \frac{1}{2}\big[\eta_{1xxxx} + \frac{1}{6}\eta_{0xxxxxx}\big]z^2 + \frac{1}{24}\eta_{0xxxxxx}\, z^4. \end{aligned} \tag{2.22}$$

Integrating equation (2.22) with the help of condition (2.15a) and then using condition (2.15b), we obtain the equation for η_2 as

$$\eta_{2tt} - \eta_{2xx} - K_2\Big[\frac{1}{2}\eta_0^2 + \Big(\int_{-\infty}^{x}\eta_{0t}dx\Big)^2\Big]_{xx} - K_1\eta_{0xxxx} - \frac{1}{45}\eta_{0xxxxxx} = 0, \tag{2.23}$$

where we have used $u_0 = -\int_{-\infty}^{x}\eta_{0t}\,dx$. Therefore, η_2 will contain terms of the form $E_2(x-t)+F_2(x-t)+tG_2(x-t)+tH_2(x-t)$ for some arbitrary functions E_2 and F_2, and the functions G_2 and H_2 dependent on functions E_0 and F_0. Since the secular term $tG_2(x-t)+tH_2(x-t)$ grows in time, η_2 will become unbounded as $t \to \infty$.

• **Equation for $\boldsymbol{\eta(x,t)}$:** Combining equations (2.17), (2.20) and (2.23) according to the series expansion (2.9), we obtain the following equation for η correct up to $O(\beta^2)$

$$\eta_{tt} - \eta_{xx} - K_2\beta^2\Big[\frac{1}{2}\eta^2 + \Big(\int_{-\infty}^{x}\eta_t\ dx\Big)^2\Big]_{xx} - K_1\beta^2\eta_{xxxx} - \frac{\beta^2}{45}\eta_{xxxxxx} = 0. \tag{2.24}$$

Since η_2 has a secular term $t(G_2(x-t)+H_2(x-t))$, the perturbation series approximation (2.9) for η is not uniformly valid for all t; but,it is valid for all $0 \le t < 1/\epsilon^2$ since η_2 is of $O(\epsilon^2)$. Therefore, for $0 \le t < 1/\epsilon^2$ equation (2.24), which is one version of the sixth-order Boussinesq equation, is appropriate for the approximate description of bi-directionally propagating small amplitude long capillary-gravity waves on the surface of shallow water for Bond number less than but very close to 1/3 (i.e., $\tau \uparrow 1/3$). Another version is introduced below.

• **Co-ordinate Transformation and Transformed Equation:** At first sight, equation (2.24) looks rather complicated. But, if we use the co-ordinate transformation

$$\left.\begin{aligned} X &= \tfrac{1}{\sqrt{K_1}}\Big(x + K_2\beta^2\textstyle\int_{-\infty}^{x}\eta(x,t)dx\Big), \\ T &= \tfrac{1}{\sqrt{K_1}}\ t, \end{aligned}\right\} \tag{2.25}$$

and substitute

$$N = \frac{3K_2}{2}\Big(\eta - K_2\beta^2\eta^2\Big), \tag{2.26}$$

then equation (2.24) is transformed into the following canonical form

$$N_{TT} - N_{XX} - \beta^2(N^2)_{XX} - \beta^2 N_{XXXX} - \epsilon_1^2\beta^2 N_{XXXXXX} = 0, \tag{2.27}$$

where we have neglected the terms of $O(\beta^3)$ and higher. Here, $\epsilon_1^2 = \frac{1}{45K_1^2}$. This equation (2.27) can be transformed to the standard form (1.2) through the following change of variables:

$$X \to \beta x, \quad T \to \beta t, \quad N \to \beta^{-2}\eta. \tag{2.28}$$

The parameter ϵ in equation (1.2) is related to the parameter ϵ_1 in equation (2.27) by $\epsilon^2 = \epsilon_1^2/\beta^2$.

3 Concluding Remarks

In this paper, the sixth-order Boussinesq equation (1.2), recently introduced by Daripa and Hua [2], is derived rigorously. It is shown here that this equation describes the bi-directional propagation of small amplitude long capillary-gravity surface waves for Bond number τ less than but very close to 1/3 (i.e., $\tau \uparrow 1/3$). Daripa and Dash [3] recently studied the sixth-order Boussinesq equation (1.2) both analytically and numerically. It is shown there that, unlike the local solitary waves, the traveling wave solutions of this equation can not vanish in the far-field. Instead, such waves must possess small amplitude fast oscillations at distances far from the core of the waves extending up to infinity. So, these solutions have the qualitative features of weakly non-local solitary wave solutions of the fifth-order KdV equation (Boyd [1], Grimshaw and Joshi [4], Hunter and Scherule [5], Pomeau et al. [7]).

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