Some useful filtering techniques for illposed problems [1]

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Abstract

Some useful filtering techniques for computing approximate solutions of illposed problems are presented. Special attention is given to the role of smoothness of the filters and the choice of time-dependent parameters used in these filtering techniques. Smooth filters and proper choice of time-dependent parameters in these filtering techniques allow numerical construction of more accurate approximate solutions of illposed problems. In order to illustrate this and the filtering techniques, a severely illposed fourth-order nonlinear wave equation is numercally solved using a three time-level finite difference scheme. Numerical examples are given showing the merits of the filtering techniques. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Illposed problems appear in many areas of practical interest such as continuum mechanics, geophysics, acoustics, electrodynamics, tomography, medicine, ecology, and various other branches of mathematical physics and mathematical analysis. Therefore, there is a considerable interest in constructing good approximate solutions of such illposed problems. One of the techniques for constructing such solutions is filtering technique. Filtering techniques attempt to construct approximate solutions of illposed problems by using selective perturbation of the initial data determined by the complete elimination or suppression of spurious errors in the short wave components of the data (cf. [5]). One of the goals of this paper is to construct and justify the use of new filtering techniques which employ smooth filters rather than sharp filters.

Filtering techniques are very popular due to their effectiveness, ease of implementation (specially, in Fourier domain) and possibility of achieving good approximate solutions of illposed partial differential equations. There are multitude of such techniques in Fourier and physical domains which can be applied to suppress dangerous short wave modes participating in numerical calculations. Techniques in physical domain involve local averaging of data and are usually not as robust as techniques in Fourier domain. Moreover, it is usually very convenient to filter in Fourier domain: take the whole data set, FFT it, multiply the FFT by a filter function and then do an inverse FFT to get back a filtered data set in physical domain. All filtering techniques in Fourier domain involve construction and application of digital filters. Use of these filtering techniques requires some guidance on the choice of filter and frequency of its application in order to achieve good approximate numerical solutions of illposed problems. This task would be much simpler if there were only one optimal filtering technique that worked best on all illposed problems. But this is not the case and it often requires a combination of analysis and experimentation to achieve the best results.

Although there has been many papers written on filtering techniques, to our knowledge there has been no systematic studies of the role that smooth filters can play in providing better approximate solutions. This is in spite of the fact that considerable research effort has been directed towards constructing better approximate solutions of illposed problems by various other techniques such as smoothing of solutions in physical domain, regularization techniques, etc. It seems appropriate to us to numerically study in some detail the effect of smoothness of filters on the accuracy of approximate numerical solutions of illposed problems. In order to do so, we construct a family of filters with varying degree of smoothness and carry out-numerical experimentation with filter being applied in a variety of ways described in the text. The filtering techniques with these filters are applied to a fourth order nonlinear illposed Boussinesq equation using a multilevel finite difference scheme. The paper is largely computational in nature. Analysis of any kind is not presented.

The paper is laid out as follows. In Section 2, we construct some smooth filter functions in the Fourier domain and discuss time-dependent parameters in these filters. An illposed Boussinesq equation and a numerical scheme for solving this equation using filtering techniques are discussed in Section 3. Section 4 discusses the algorithms: the ways the filters are applied to the multi-level finite difference scheme. Section 5 presents some relevant numerical results obtained with various filters. Concluding remarks are made in Section 6.

2. Filtering techniques

2.1. Description of the filters

High wave number modes of the Fourier spectrum of the initial data with amplitudes comparable to or less than roundoff error are naturally most contaminated by roundoff error. Catastrophic growth of roundoff errors in these modes due to short-wave instability causes significant contamination of numerical solutions as these solutions evolve in time. Filtering techniques attempt to eliminate such spurious effects of roundoff error on the numerical solutions by using appropriate filtering techniques. There are multitude of filters and the simplest among these is probably the sharp filter whose value is either one or zero depending on whether the wave number is below or greater than some cutoff wave number, n_c. Even though sharp filter has been used successfully for illposed problems, smooth filters

(obtained by smoothing of the sharp vertical edge at the cutoff wave number of the sharp filter) with appropriate choice of time-dependent filter parameters, when applied properly, can provide even better accuracy of the numerical solutions. In order to exemplify this issue, we first define these filters of varying degree of smoothness which are then used to generate solutions of an illposed nonlinear wave equation. Numerical solutions appear to be the best for certain choices of time dependent parameters which are presented in the Section 5.

Here and below, the Fourier modal amplitudes of a numerical solution before and after the use of a filter $\Phi(n)$ are denoted by a_n and $\tilde{a}_n$, respectively where n is the wave number and

$$\tilde{a}_n = a_n \Phi(n). \tag{2.1}$$

All filters below are defined for positive wave numbers and their values for negative wave numbers are obtained from the property $\Phi(-n) = \Phi(n)$ since the data to be subjected to filtering is real. Each of the filters discussed below contains certain parameters which are time dependent. This time dependence is not shown explicitly in (2.1) for notational convenience.

We experimented with following four filters of increasing order of smoothness. The sharp filter $\Phi_1(n)$ is defined as

$$\Phi_1(n) = \begin{cases} 1, & n \leqslant n_c, \\ 0, & n > n_c, \end{cases} \tag{2.2}$$

where n_c is a cutoff wave number and is chosen carefully based on the computational noise level which is largely determined by the machine representation of numerical solutions Fourier spectrums or equivalently machine roundoff error.

Below we define three smooth filters of family Φ_3, namely $\Phi_3^1(n)$, $\Phi_3^2(n)$, and $\Phi_3^3(n)$ which are C^0, C^2 and C^4, respectively. These filter functions are constructed by smoothing the vertical edge of the sharp filter $\Phi_1(n)$ over $(n_2 - n_c)$ points. The choice of $n_2 > n_c$ is made clear later. Filters Φ_3^j, $j = 1, 2, 3$ are defined as follows:

$$\Phi_3^j(n) = \begin{cases} 1, & n \leqslant n_c, \\ 1 - g_j(\hat{n}), & n_c < n < n_2, \\ 0, & n \geqslant n_2, \end{cases} \tag{2.3}$$

where

$$\hat{n} = \frac{n - n_c}{n_2 - n_c}, \tag{2.4}$$

and

$$g_1(x) = x, \quad 0 < x < 1, \tag{2.5}$$

$$g_2(x) = \begin{cases} \frac{9}{2}x^3, & 0 < x \leqslant \frac{1}{3} \\ 9x^3 + \frac{27}{2}x^2 - \frac{9}{2}x + \frac{1}{2}, & \frac{1}{3} < x \leqslant \frac{2}{3} \\ 1 - \frac{9}{2}(1 - x)^3, & \frac{2}{3} < x < 1. \end{cases} \tag{2.6}$$

$$g_3(x) = \begin{cases} \frac{625}{24}x^5, & 0 < x \leqslant \frac{1}{5}, \\[2mm] -\frac{625}{6}x^5 + \frac{3125}{24}x^4 - \frac{625}{12}x^3 + \frac{125}{12}x^2 \\ \quad -\frac{25}{24}x + \frac{1}{24}, & \frac{1}{5} < x \leqslant \frac{2}{5}, \\[2mm] \frac{625}{4}x^5 - \frac{3125}{8}x^4 + \frac{4375}{12}x^3 - \frac{625}{4}x^2 \\ \quad +\frac{775}{24}x - \frac{21}{8}, & \frac{2}{5} < x \leqslant \frac{3}{5}, \\[2mm] 1 + \frac{625}{6}(1-x)^5 - \frac{3125}{24}(1-x)^4 + \frac{625}{12}(1-x)^3 \\ \quad -\frac{125}{12}(1-x)^2 + \frac{25}{24}(1-x) - \frac{1}{24}, & \frac{3}{5} < x \leqslant \frac{4}{5}, \\[2mm] 1 - \frac{625}{24}(1-x)^5, & \frac{4}{5} < x < 1. \end{cases} \tag{2.7}$$

The functions $g_1(x)$, $g_2(x)$ and $g_3(x)$ are C^0, C^2 and C^4 functions, respectively.

It should be noted that after application of any of these filters, all Fourier modes with wave numbers greater than n_c for the case of filter Φ_1 and n_2 for the case of filters of family Φ_3 have zero amplitudes. The relative merits of these filters are tested by experimenting with these filters on an illposed nonlinear wave equation which is discussed in Section 3.

Next we discuss the choice of the parameters n_c and n_2. The choice of these parameters as it is currently practiced has a strong empirical basis, and there are generally tradeoffs involved between different choices of these parameters in such a way that there is usually no general agreement on the best choice of these parameters. There is no exact science for its selection and a suitable choice of these parameters may require some numerical experimentation.

2.2. Choice of n_c and n_2

For our purposes here, we denote the parameter n_c as the smallest wave number with amplitude $|a_{n_c}| = 10^{-m}$ where the filter level, m, is to be chosen carefully based on roundoff error and severity of the growth rate of short waves. Since high wave number modes whose amplitudes are less than or of the order of roundoff error are most significantly contaminated initially, it appears sensible to choose these parameters based on roundoff error so that these dangerous modes are considerably suppressed. This guiding principle appears to be sufficient for 7-digit arithmetics (single precision) calculation. For our single precision calculations in this paper, we have chosen after some experimentation filter level $m = 5$ so that $|a_{n_c}| = 10^{-5}$ which is higher than the roundoff error 10^{-7}. Fig. 1 depicts such a choice of n_c for a discrete Fourier spectrum where no mode may have an amplitude exactly equal to 10^{-5}.

Two points are worth mentioning here. Firstly, parameters n_2 and n_c are time-dependent since their choice at any particular time level is based on the Fourier spectrum of the numerical solution at that time level only. Secondly, the filter level in higher precision calculations may have to be significantly higher than that allowed by the machine roundoff error in order to avoid very short waves that may otherwise be very dangerous due to growth of truncation errors in these short waves, specially for severely illposed problem as the one discussed in Section 3. However, such considerations are not of importance for our single precision calculations in this paper and one may refer [2] for such high precision calculations.
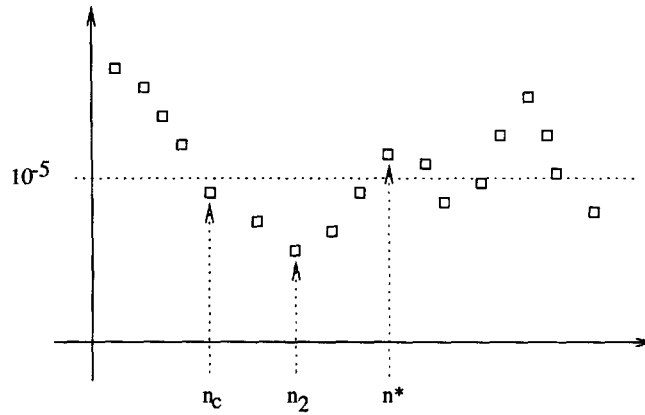
Fig. 1. The positions of the points n_c, n_2 and n^* we used in the algorithm. The plot shows a hypothetical Fourier spectrum (Fourier modal amplitudes against wave numbers).

We numerically experimented with application of the above filters on a severely illposed problem discussed in the next section. Particular attention was paid to the effect of frequency of applications of these filters (i.e. at every time level to intervals of few time levels) on the accuracy of the numerical solutions. Best results were not obtained with application of filters at every time level. Rather, best results were obtained when filter was applied only if the amplitude of one or more of the modes with wave numbers $n > n_c$ exceeded the filter level (see Fig. 1) and the parameter n_2 ($n_c < n_2 < n^*$) was chosen so that the Fourier mode with wave number n_2 has the smallest amplitude in this wave number range ($n_c < n_2 < n^*$). Next we discuss application of these filters in solving an illposed Boussinesq equation.

3. Illposed Boussinesq equation

Filtering techniques are applied to solve the following illposed nonlinear wave equation

$$u_{tt} = (p(u))_{xx} + u_{xxxx}, \tag{3.1}$$

where $p(u) = u + u^2$. This equation describes propagation of long waves in shallow water under gravity [6], in one-dimensional nonlinear lattices and in nonlinear strings [7]. This equation is chosen for following reasons:

- It has exact traveling wave solutions whose Fourier modal amplitudes do not change in time. This facilitates setting up of numerical experiments so that the effect of short wave instability and the performance of various filtering techniques can be discerned very easily.
- This equation has very severe short wave instability which is useful in testing the robustness of various filtering techniques.
- The numerical scheme for solving this equation requires using solutions at two previous time levels. This provides some extra freedom in choosing the filter parameters at both of these time levels as will be seen later.

• This equation is so severely illposed that this equation is not well understood. There are analytical and numerical difficulties in studying this equation. ([3, 2]). Hopefully, filtering techniques explored here will be found helpful in providing better understanding of this equation.

Eq. (3.1) is linearly illposed as the corresponding linearized pde has decaying as well as growing modes, $e^{\sigma t + ikx}$, with the dispersion relation about the constant state, u_c, given by

$$\sigma_{\mp} = \mp k\sqrt{k^2 - p'(u_c)}. \tag{3.2}$$

Thus equilibrium states in the elliptic region (i.e. $p'(u_c) = 1 + 2u_c < 0$) are unstable to all modes and states in the hyperbolic region are unstable to modes $|k| > \sqrt{p'(u_c)}$. Since growth rate, i.e. the real part of σ_+, is a monotonically increasing function of wave number, there is no wave number with maximal rate of stability. It follows from the dispersion relation (3.2) that

$$\sigma \approx k^2 \quad \text{as} \quad k \to \infty. \tag{3.3}$$

This short-wave instability causes severe sensitivity of the solutions to small errors in high wave number modes. Moreover the growth rate of short waves here is so severe that this equation's solutions may not exist in classical sense for arbitrary initial data. However, solutions exist for special choices of initial data. For example, this equation allows unidirectional (solitary wave) as well as bidirectional waves ([3, 4, 6]). To examine and compare the performance of various filters discussed in the preceding section, exact solitary wave solution

$$u^s(x,t) = A \operatorname{sech}^2\{\sqrt{A/6}(x - ct)\} \tag{3.4}$$

is considered where A is the amplitude of the solitary wave and $c = \mp \sqrt{2A/3}$ is the speed of the solitary wave.

Eq. (3.1) is solved numerically in a finite domain, $a \leqslant x \leqslant b$ using the following finite difference method with uniform grid spacings h in x and τ in t. Using v_j^n to denote the approximated value of $u(x,t)$ at $x = a + jh$, $t = n\tau$ and using usual finite difference operators D^+ and D^- to denote forward and backward differences, Eq. (3.1) is approximated by

$$\frac{D_t^+ D_t^- v_j^n}{\tau^2} = \frac{D_x^+ D_x^-(p(v_j^n))}{h^2} + \frac{(D_x^+ D_x^-)^2(v_j^{n+1} + v_j^{n-1})}{2h^4}, \tag{3.5}$$

for $\tau > 0$ and $0 < j < N$ ($0 \leqslant j < N$ in case of periodic domain) where $b - a = Nh$. The truncation error is $E(h,\tau) = O(h^2) + O(\tau^2)$. We use following fourth-order accurate formulae [1] to estimate boundary conditions $v(a - h, n\tau)$ and $v(b + h, n\tau)$, for $n \geqslant 0$.

$$v(a - h, n\tau) = -\tfrac{3}{2}v(a,n\tau) + 3v(a + h, n\tau) - \tfrac{1}{2}v(a + 2h, n\tau) - 3v'(a, n\tau)h, \tag{3.6}$$

$$v(b + h, n\tau) = -\tfrac{3}{2}v(b,n\tau) + 3v(b - h, n\tau) - \tfrac{1}{2}v(b - 2h, n\tau) + 3v'(b, n\tau)h, \tag{3.7}$$

and the following third-order accurate initialization to estimate $v(jh, \tau)$, $0 \leqslant j \leqslant N$,

$$v(.,\tau) = v(.,0) + v'(.,0)\tau + v''(.,0)\tfrac{\tau^2}{2} + O(\tau^3), \tag{3.8}$$

where $v(.,\tau)$ and $v'(.,\tau)$ are given, $v''(.,\tau)$ can be obtained directly from using the Boussinesq Eq. (3.1). The dispersion relation for the above finite difference scheme for time step size $\tau = 0.01$ and spatial grid size $h = 0.5$ compare very well with that of the exact Boussinesq Eq. (3.1) (see [2]). These mesh sizes are used for numerical results presented in the next section.

4. The Algorithms

The numerical method based on finite difference equation (3.5) and the filtering criterion of Section 2 is referred below as *Algorithm*-I.

The numerical solution v^{n+2} depends on previous two time levels' solutions v^n and v^{n+1}, both of which may not participate in the filtering process in the Algorithm-I. It is observed during numerical experimentation that if v^{n+1} participates in the filtering process and v^n does not, then filtering v^n with the parameters n_2 and n_c chosen based on the solution v^{n+1} can improve the accuracy of the numerical solution v^{n+2} considerably. Application of the numerical method based on finite difference Eq. (3.5) and this modified form of filtering is referred below as *Algorithm*-II. This algorithm ensures that whenever the filter is applied on v^{n+1}, it will be applied on v^n if it did not participate in the filtering process during previous time step.

5. Numerical results

Exact solitary wave solution (3.4) of Eq. (3.1) travels at a constant speed and therefore its modal amplitudes (i.e. the absolute values of its Fourier coefficients) do not depend on time. This time independence of the modal amplitudes of a traveling solitary wave can be used very effectively to investigate the performance of various filtering techniques. Therefore numerical calculations are performed with initial data derived from exact solitary wave solution given by (3.4) with amplitude $A = 0.5$.

Oscillations in numerical solutions start developing due to severe short wave instability as soon as the high wave number modal amplitudes exceed an approximate value of 10^{-5} which happens at a time level $t = 0.6$. Fig. 2 shows logarithm of the amplitudes of participating Fourier modes of the exact solitary wave and numerical solutions at two time levels ($t = 0.6$ and 1.0) against their wave numbers. It is seen here that high wave number participating modes of the initial condition whose amplitudes are less than the roundoff error are contaminated by machine roundoff error. These errors are amplified severely by the short wave instability and in time tend to contaminate even the low wave number modes of the spectrum due to nonlinearity.

In order to improve the calculations, filtering technique is applied here with sharp filter Φ_1 as well as smooth filters of the family Φ_3. As per filtering criterion discussed in Section 2, numerical solutions for $t < 0.6$ did not require to be filtered. Filtering technique is applied for the first time at a time level $t = 0.6$. Fig. 3 shows the effect of these filters on the Fourier spectrum of the solution at this time level ($t = 0.6$) with filters' parameter n_c (see Eqs. (2.2) and (2.3), also Fig. 3) chosen based on the threshold level $|a_{n_c}| = 10^{-5}$. Since Fourier spectrum is symmetric, it is only necessary to show the right half of the spectrum in Fig. 2. All Fourier modes with wave numbers greater than n_c for the case of filter Φ_1 and n_2 for the case of filter Φ_3^2 have zero amplitudes after the application of the filters according to Eqs. (2.2) and (2.3). Zero amplitudes of these modes could not be shown in Fig. 3 because zero is not in the range of the ordinates of the plots in Fig. 3.

We have done extensive numerical experiments with the filters mentioned above and also tested the appropriateness of applying the filters at various time levels. It is found that application of a filter at every time step deteriorates the accuracy of the numerical solutions. Best result is obtained with filters being applied only when it is necessary, i.e. when one or more of the high-wave number modal amplitudes exceed a value approximately 10^{-5} as discussed earlier.
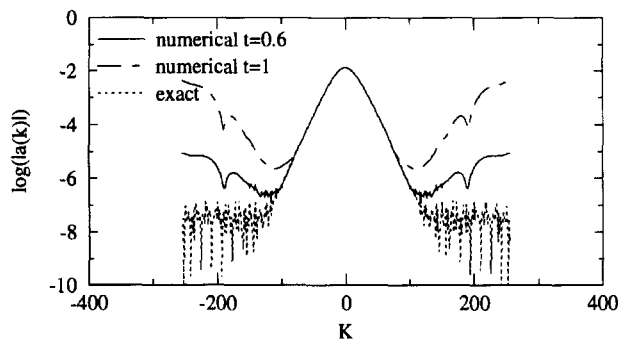
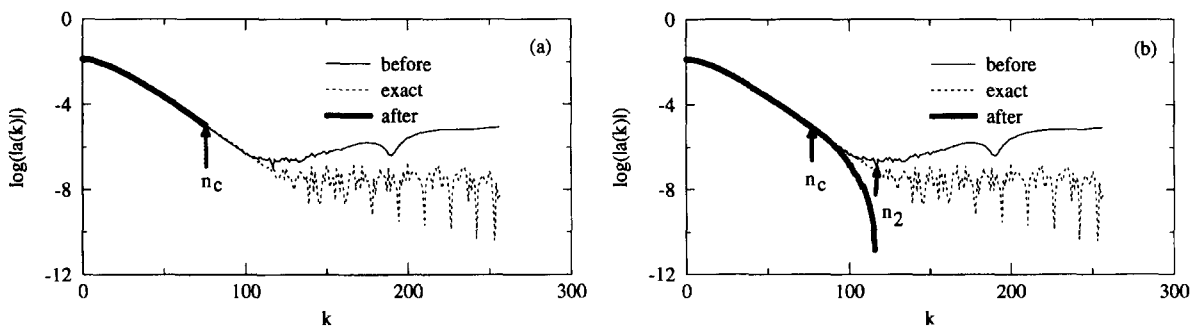Fig. 2. $\log_{10}|a_k|$ it vs./ k where $|a_k|$ is the amplitude of the Fourier mode with wave number k.

Fig. 3. Effect of the filters (a) Φ_1 and (b) Φ_3^2 in suppressing spurious growth of roundoff errors. Fourier spectrums of the numerical solutions at $t = 0.6$ before and after the use of the filters are shown. The Fourier spectrum of the exact solution is also shown here for comparison purposes.

Table 1
The L_2 and L_∞ error estimates of numerical solutions using filters Φ_1, Φ_3^1, Φ_3^2, and Φ_3^3 with the algorithm-I

Time t	Φ_1		Φ_3^1		Φ_3^2		Φ_3^3	
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.6	$7.34 \cdot 10^{-4}$	$2.49 \cdot 10^{-4}$	$3.62 \cdot 10^{-4}$	$1.48 \cdot 10^{-4}$	$3.50 \cdot 10^{-4}$	$1.37 \cdot 10^{-4}$	$3.49 \cdot 10^{-4}$	$1.35 \cdot 10^{-4}$
1	$2.75 \cdot 10^{-2}$	$4.45 \cdot 10^{-3}$	$5.58 \cdot 10^{-3}$	$9.02 \cdot 10^{-4}$	$5.50 \cdot 10^{-3}$	$8.69 \cdot 10^{-4}$	$5.50 \cdot 10^{-3}$	$8.67 \cdot 10^{-4}$
2	0.11	$1.55 \cdot 10^{-2}$	$4.52 \cdot 10^{-2}$	$7.95 \cdot 10^{-3}$	$4.70 \cdot 10^{-2}$	$9.05 \cdot 10^{-3}$	$4.60 \cdot 10^{-2}$	$8.07 \cdot 10^{-3}$
3			0.18	$2.87 \cdot 10^{-2}$	0.18	$2.87 \cdot 10^{-2}$	0.18	$2.86 \cdot 10^{-2}$

Instead of inundating this paper with lots of results, we summarize the main points of the results of numerical experiments in Tables 1 and 2 and Fig. 4. The L_2 and L_∞ errors of numerical solutions obtained with various filters using Algorithm-I and Algorithm-II are tabulated in Tables 1 and 2, respectively. The main observations from these tables are the following.

• Both algorithms give more accurate numerical solutions with smooth filters of family Φ_3 than with the sharp filter Φ_1. But there are no big differences among Φ_3 filters, i.e. the degree of smoothness of smooth filters do not affect the solutions significantly in either algorithms.

Table 2

The L_2 and L_∞ error estimates of numerical solutions using filters Φ_1, Φ_3^1, Φ_3^2, and Φ_3^3 with the algorithm-II

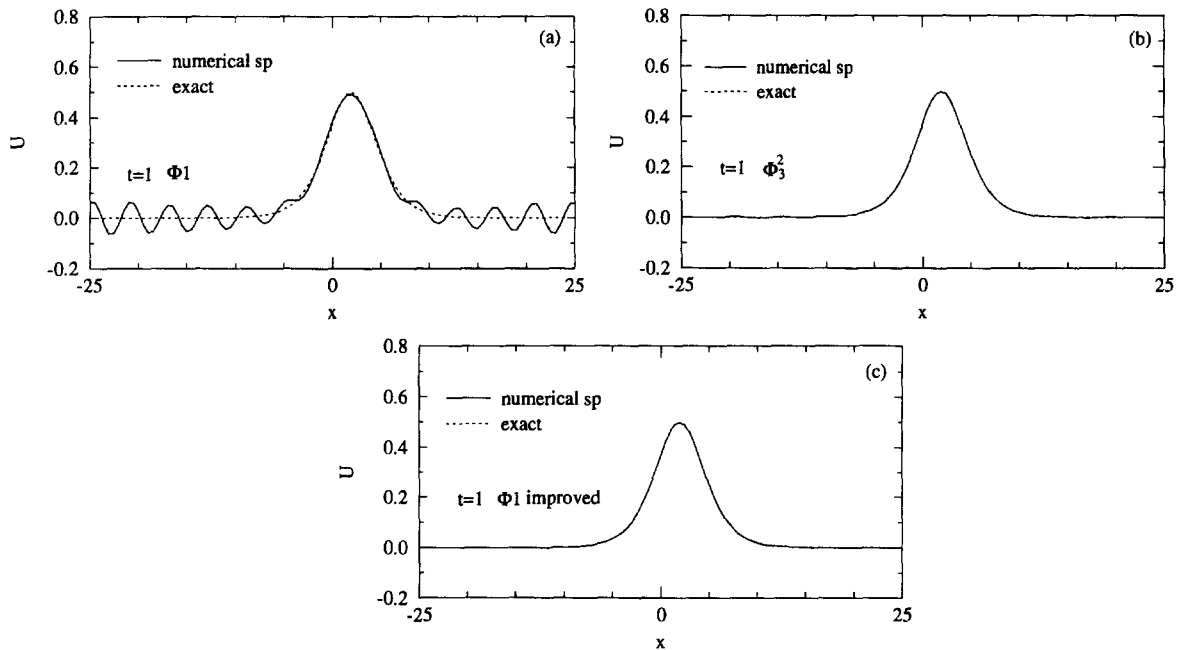| Time t	Φ_1		Φ_3^1		Φ_3^2		Φ_3^3	
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.6	$7.34 \cdot 10^{-4}$	$2.49 \cdot 10^{-4}$	$3.62 \cdot 10^{-4}$	$1.48 \cdot 10^{-4}$	$3.50 \cdot 10^{-4}$	$1.37 \cdot 10^{-4}$	$3.49 \cdot 10^{-4}$	$1.35 \cdot 10^{-4}$
1	$1.95 \cdot 10^{-3}$	$8.83 \cdot 10^{-4}$	$1.02 \cdot 10^{-3}$	$3.53 \cdot 10^{-4}$	$1.00 \cdot 10^{-3}$	$3.41 \cdot 10^{-4}$	$1.00 \cdot 10^{-3}$	$3.37 \cdot 10^{-4}$
2	$1.40 \cdot 10^{-2}$	$4.79 \cdot 10^{-3}$	$4.90 \cdot 10^{-3}$	$1.37 \cdot 10^{-3}$	$4.86 \cdot 10^{-3}$	$1.35 \cdot 10^{-3}$	$5.10 \cdot 10^{-3}$	$1.50 \cdot 10^{-3}$
3	0.36	0.15	$1.20 \cdot 10^{-2}$	$3.14 \cdot 10^{-3}$	$1.42 \cdot 10^{-2}$	$4.43 \cdot 10^{-3}$	$4.41 \cdot 10^{-2}$	$8.70 \cdot 10^{-3}$
4			0.12	$4.44 \cdot 10^{-2}$	0.15	$5.93 \cdot 10^{-2}$	0.17	$6.44 \cdot 10^{-2}$

Fig. 4. Comparison of numerical and exact solutions at the time level $t = 1$ using different filtering techniques: (a) Φ_1 filter with the algorithm-I; (b) Φ_3^2 filter with the algorithm-I; and (c) Φ_1 filter with the algorithm-II. The calculations were done in single precision (7 digit arithmetics) with $h = 0.5$ and $\tau = 0.01$.

- Numerical results with Algorithm-II are more accurate than that with Algorithm-I for a fixed choice of filter.
- If there were a choice between switching to Algorithm-II from Algorithm-I and switching to a smooth filter from the sharp filter in Algorithm-I, the latter is better than the former.

We illustrate the effects of the sharp and smooth filters on the numerical solutions in Fig. 4 by comparing these against the exact solution given by (3.4) at a time level $t = 1.0$. Numerical results in Figs. 4(a) and (b) have been obtained using Algorithm-I with the sharp filter Φ_1 and the smooth filter Φ_3^2, respectively. Differences in the numerical solutions with smooth filters of family Φ_3 are almost identical (see Tables 1 and 2) and therefore we have shown numerical results with only one
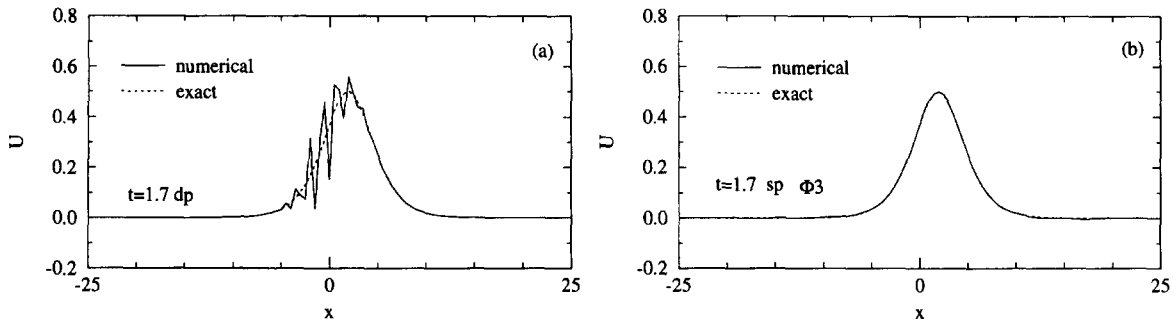
Fig. 5. Comparison of numerical and exact solutions at the time level $t = 1.7$. The calculations were done in (a) *double precision* (15 digit arithmetics) without any use of filter; (b) *single precision* (7 digit arithmetics) with filter Φ_3^2 at filter level 5. All the computations use $h = 0.5$ and $\tau = 0.01$.

of these smooth filters in Fig. 4(b). It is clearly seen here that the accuracy of the numerical solution with the smooth filter is far better than that with the sharp filter.

Next we illustrate the effect of using Algorithm-II instead of Algorithm-I in Fig. 4(c). This figure compares the numerical solution at $t = 1.0$ obtained using the sharp filter Φ_1 and Algorithm-II with the exact solution. A comparison of this with the result of Fig. 4(a) shows that accuracy of the numerical solution, even with the sharp filter, can be improved considerably using Algorithm-II instead of Algorithm-I. A comparison of Figs. 4(b) and (c) which compare favorably with each other within resolution of the plots should not give the impression that sharp filter Φ_1 in Algorithm-II performs as good as the smooth filter Φ_3^2 in Algorithm-I. This may be true at earlier times as this figure seems to indicate but this is definitely the case at later times as is evident from the error estimates in Tables 1 and 2.

These and other numerical experiments strongly suggest that smooth filters should be preferred over sharp filters for more accurate solutions of illposed problems. In fact, numerical solutions even in lower precision calculations with smooth filters can give much more accurate solutions than higher precision calculation without filters. In other words, smooth filters is more effective than high precision arithmetic in providing more accurate solutions. Numerical results in Fig. (5) which were obtained using Algorithm-I show this. In each of Figs. 5(a) and (b), numerical results at $t = 1.7$ are compared against the exact solution. Numerical result with sharp filter is not shown as this is much worse than that with smooth filters which can be inferred directly from Table 1 and has been discussed earlier in this section.

6. Conclusion

Below we summarize our main points based on our extensive numerical experiments with various filters on a severely illposed problem. These findings may have significant bearing on devising effective filtering techniques for other illposed problems as well.

- Filtering techniques should use smooth filters in preference to sharp filter whenever possible. Smooth filters when applied properly yield better approximate solutions than sharp filter. However, degree of smoothness of the filter does not seem to affect the performance of the filter in any significant way.

- In order to enable better approximate solutions in multilevel finite difference scheme, it may be necessary to filter numerical solutions at all previous time levels that participate in the scheme depending on whether the numerical solution at most recent time level that is being used has been filtered. This is a more general statement than what we have shown here. In this paper, we have shown this to be the case for three time level finite difference scheme (i.e. Algorithm-II is better than Algorithm-I).

Acknowledgements

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