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Weakly non-local solitary wave solutions of a singularly perturbed Boussinesq equation

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Abstract

We study the singularly perturbed (sixth-order) Boussinesq equation recently introduced by Daripa and Hua [Appl. Math. Comput. 101 (1999) 159]. This equation describes the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for bond number less than but very close to 1/3. On the basis of far-field analyses and heuristic arguments, we show that the traveling wave solutions of this equation are weakly non-local solitary waves characterized by small amplitude fast oscillations in the far-field. Using various analytical and numerical methods originally devised to obtain this type of weakly non-local solitary wave solutions of the singularly perturbed (fifth-order) KdV equation, we obtain weakly non-local solitary wave solutions of the singularly perturbed (sixth-order) Boussinesq equation and provide estimates of the amplitude of oscillations which persist in the far-field. © 2001 IMACS. Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we study the singularly perturbed (sixth-order) Boussinesq equation

$$\eta_{tt} = \eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx} + \epsilon^2\eta_{xxxxxx} \tag{1}$$

where ϵ is a small parameter. This equation was originally introduced by Daripa and Hua [1] as a regularization of the ill-posed classical (fourth-order) Boussinesq equation which corresponds to $\epsilon = 0$ in Eq. (1). It is well-known that the fourth-order Boussinesq equation possesses the traveling-solitary-wave solutions (see [2,3]).

The physical relevance of Eq. (1) in the context of water waves was recently addressed by Dash and Daripa [4]. It was shown that this equation actually describes the bi-directional propagation of small

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amplitude and long capillary-gravity waves on the surface of shallow water for bond number (surface tension parameter) less than but very close to 1/3. So, it is closely related to the singularly perturbed (fifth-order) KdV equation originally derived by Hunter and Scherule [5] which is restricted only to uni-directional propagating waves.

The fifth-order KdV equation has been studied extensively over last decade. It has been shown that the traveling wave solutions of this equation do not vanish at infinity. Instead, they possess small amplitude fast oscillations at infinity (e.g. [5–15]). These are well-known as weakly non-local solitary waves. These types of solutions are also known to exist for full non-linear water wave problem for positive bond number less than 1/3 (e.g. [16–19]) and internal waves in stratified fluid for mode number greater than one (e.g. [20,21]). The amplitude of oscillations associated with the weakly non-local solitary wave solutions of the fifth-order KdV equation was estimated analytically by Akylas and Yang [6], Grimshaw and Joshi [14] and Pomeau et al. [15], and numerically by Benilov et al. [9] and Boyd [10].

In this paper, we construct the weakly non-local solitary wave solutions of the sixth-order Boussinesq Eq. (1) in the form of traveling waves by using analytical and numerical methods originally devised to obtain this type of weakly non-local solitary wave solutions of the fifth-order KdV equation. We also obtain the estimates of the amplitude of the oscillatory tails associated with these weakly non-local solitary waves.

2. Analysis of the problem

Since Eq. (1) has solitary wave solutions for $\epsilon = 0$, the natural question arises whether Eq. (1) also admits solitary wave solutions for small values of ϵ. Therefore, we seek a traveling wave solution of Eq. (1) in the form $\eta(x,t) = \eta(x-ct)$, where c is the phase speed (velocity) of the wave. Substituting it in Eq. (1) and using x for the new variable $x-ct$ yields

$$(1-c^2)\eta_{xx} + (\eta^2)_{xx} + \eta_{xxxx} + \epsilon^2\eta_{xxxxxx} = 0 \tag{2}$$

The question now becomes whether Eq. (2) admits solutions which decay exponentially to zero as $x \to \pm\infty$ for any small positive value of ϵ. Since we are interested in bounded solutions of Eq. (2) as $x \to \pm\infty$, on integrating Eq. (2) twice and taking the constants of integration as zero, we obtain

$$(1-c^2)\eta + \eta^2 + \eta_{xx} + \epsilon^2\eta_{xxxx} = 0 \tag{3}$$

It can be easily shown that an approximate solution of Eq. (3) can be obtained as a regular asymptotic expansion in ϵ^2 in the form

$$\eta = \eta_0 + \epsilon^2(-10\gamma^2\eta_0 + \tfrac{5}{2}n_0^2) + \cdots \tag{4}$$

where $\eta_0 = 6\gamma^2\,\mathrm{sech}^2(\gamma x)$ is the solitary wave solution of the fourth-order Boussinesq equation and γ free parameter characterizing the width of the wave. The phase speed c is related to γ by $c^2-1 = 4\gamma^2 + 16\epsilon^2\gamma^4 + \mathrm{O}(\epsilon^4)$. It is to be noted here that the expansion (Eq. (4)) can be continued to any arbitrarily higher order. The general nth term in the series (Eq. (4)) will be an nth order polynomial in η_0. Since η_0 is symmetric about $x=0$ and decays down to zero exponentially as $x \to \pm\infty$, the form of solution (Eq. (4)) implies that η will also be symmetric about $x=0$ and will decay down to zero exponentially as $x \to \pm\infty$. So, by the method of regular asymptotic analysis, we only get exponentially decaying solution in the far-field. However, as we will see below, the far-field analysis contradicts this.

If we assume that η is small in the far-field $x \to \pm\infty$, then Eq. (3) linearizes to

$$(1-c^2)\eta + \eta_{xx} + \epsilon^2\eta_{xxxx} = 0 \quad \text{as} \quad x \to \pm\infty \tag{5}$$

Eq. (5) has solutions of the form $\eta = \exp(\mathrm{i}px)$ provided $\epsilon^2 p^4 - p^2 = (c^2-1)$. Since $|c| > 1$, this characteristics equation has two real roots (which correspond to the oscillatory behavior of η at infinity) and two purely imaginary roots (which correspond to decaying and growing behavior of η at infinity). For a local solitary wave, only the root which corresponds to the decaying behavior of η at infinity is acceptable. This then implies the necessity of three independent boundary conditions on η as $x \to \infty$, with three more as $x \to -\infty$, leading altogether to the necessity of six independent boundary conditions on η for a fourth-order differential Eq. (3). Therefore, we cannot force η to vanish at both $x \to \infty$ and $x \to -\infty$. There will be an oscillatory behavior at least on one side at infinity with the general form given by

$$\eta = A_\pm \sin\left[\frac{q}{\epsilon}(x+\phi_\pm)\right] \quad \text{as} \quad x \to \pm\infty \tag{6}$$

where $q^2 = \epsilon^2 p_{\mathrm{Re}}^2 = 1 + \epsilon^2(c^2-1) + \mathrm{O}(\epsilon^4) = 1 + 4\epsilon^2\gamma^2 + \mathrm{O}(\epsilon^4)$, where p_{Re} is the real root of the characteristics equation. Here $A_\pm$ and $\phi_\pm$ are the amplitude and phase shift constant of the oscillatory tails as $x \to \pm\infty$. For symmetric weakly non-local solitary wave solutions, $A_+ = A_- = A$ and $\phi_+ = \phi_- = \phi$. It is to be noted that the frequency of oscillations $(q/\epsilon) \to (1/\epsilon)$ as $\epsilon \to 0$, and hence, the far-field oscillations are very fast. In the following sections, we will obtain estimates of the amplitude A of the tail oscillations with $\phi = 0$ both analytically and numerically.

3. Perturbation analysis in the complex plane

In this section, we will construct the oscillatory tails and estimate their amplitude by extending the problem into the complex plane and using a perturbation analysis in the complex plane as in Grimshaw and Joshi [14], Kruskal and Segur [22] and Pomeau et al. [15]. This method is well-known as the technique of asymptotics beyond all orders. We will see that the amplitude of tail oscillations is exponentially small that lies beyond all orders in the regular asymptotic expansion of form (Eq. (4)) for the solution η.

Since $\eta_0(x)$ is singular in the complex x-plane at $x = \pm(2n+1)(\mathrm{i}\pi/2\gamma)$, $n = 0, 1, 2, \ldots$, the core solution $\eta(x)$ given by Eq. (4) cannot describe the actual behavior of the solution of Eq. (3) in the neighborhood of these singular points. In fact, the perturbation term $\epsilon^2\eta_{xxxx}$ cannot be considered as of lower order than the other terms in Eq. (3) in the neighborhood of these singular points. So, it is important to consider the solution structure of Eq. (3) near these singular points. To do this, we need to consider a rescaling through which the small parameter ϵ^2 is removed from the highest derivative term in Eq. (3). This problem is called the inner problem.

We consider the singularity closest to the real axis in the upper half-plane. We introduce the inner variables y and η_{i} defined by $x = (\mathrm{i}\pi/2\gamma) + \epsilon y$ and $\eta_{\mathrm{i}} = \epsilon^2\eta$, where the subscript i refers to the inner problem. Substituting it in Eq. (3) and neglecting the term containing the small parameter ϵ^2, we obtain the inner problem as

$$\eta_{\mathrm{i}}^2 + \eta_{\mathrm{i}yy} + \eta_{\mathrm{i}yyyy} = 0 \tag{7}$$

To find the solution of the original problem (Eq. (3)), we need to solve the inner problem (Eq. (7)) and connect the asymptotic behavior of the inner solution at large distances to that of the core (outer) solution (Eq. (4)) by matching their asymptotic behaviors in a region where they both make sense.

To the leading order, the asymptotic behavior of η_0 near the singularity $\mathrm{i}\pi/2\gamma$ is given by $\eta_0 = -6\gamma^2 \operatorname{cosech}^2(\gamma\epsilon y) \approx -6/(\epsilon y)^2$ as $\epsilon y \to 0$. Therefore, to the leading order, the asymptotic behavior of the outer solution (Eq. (4)) near the singularity $\mathrm{i}\pi/2\gamma$ will be given by

$$\eta \approx \frac{1}{\epsilon^2}\left(-\frac{6}{y^2}+\frac{90}{y^4}\right) \quad \text{as} \quad \epsilon y \to 0 \tag{8}$$

Hence, it should be matched to a solution of the inner problem with the asymptotic behavior.

$$\eta_{\mathrm{i}} \approx \frac{6}{y^2}+\frac{90}{y^4} \quad \text{as} \quad |y| \to \infty \quad \text{or} \quad |y| \gg 1 \tag{9}$$

In view of Eq. (9), the solution of the inner problem (Eq. (7)) is constructed as an asymptotic series in $1/y^2$ of the form

$$\eta_{\mathrm{i}} \approx -\frac{6}{y^2}+\frac{90}{y^4}+\sum_{n=3}^{\infty}\frac{a_n}{y^{2n}} \quad \text{as} \quad |y| \to \infty \quad \text{or} \quad |y| \gg 1 \tag{10}$$

When Eq. (10) is substituted into Eq. (7), the coefficients of $y^{-(2n+4)}$ give

$$(2n-2)(2n-1)(2n)(2n+1)a_{n-1} + (2n+4)(2n-3)a_n + \sum_{k=2}^{n-1} a_k a_{n+1-k} = 0 \quad \text{for} \quad n \geq 3 \tag{11}$$

with $a_1 = -6$ and $a_2 = 90$. So, a_n can be obtained from Eq. (11) recursively. As $n \to \infty$, the non-linear term in Eq. (11) becomes less important. Therefore, an asymptotic formula for a_n correct up to $\mathrm{O}(1/n^2)$ is given by

$$(2n-2)(2n-1)(2n)(2n+1)a_{n-1} + (2n+4)(2n-3)a_n \approx 0 \quad \text{for large } n \tag{12}$$

Eq. (12) recursively gives

$$a_n \approx \frac{(2n+1)(2n-1)}{(2n+2)(2n+4)}(-1)^n(2n-1)!K \quad \text{for large } n \tag{13}$$

where K is some constant. The value of K is obtained by computing the exact values of a_n from Eq. (11) for some large values of n and matching it with the asymptotic formula (Eq. (13)). The value of K was found to be 59.91.

With the coefficients a_n given by Eq. (11) for all $n \geq 3$ and by Eq. (12) or Eq. (13) for large n, the asymptotic series solution (Eq. (10)) of the inner problem (Eq. (7)) diverges for all y. However, it can still be summed using the method of Borel summation [23]. So, we express $\eta_{\mathrm{i}}(y)$ in the form of a Laplace transform (also see [14,15]) given by

$$\eta_{\mathrm{i}}(y) = \int_0^\infty V\frac{p}{y}\mathrm{e}^{-p}\,\mathrm{d}p = y\int_0^\infty V(s)\mathrm{e}^{-sy}\,\mathrm{d}s = \int_0^\infty V'(s)\mathrm{e}^{-sy}\,\mathrm{d}s \tag{14}$$

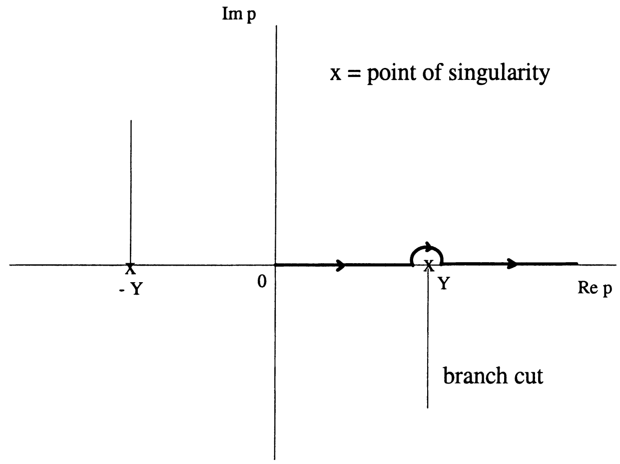

Fig. 1. Deformation of the integration path around the branch cut of the singularity at $p = Y$ in a clockwise direction. The point of singularity $p = Y$ lies on the real axis in Re, $p > 0$.

where $V(s)$ is an unknown function with $V(0) = 0$, and $V'(s)$ denotes the derivative of $V(s)$ with respect to s. The integration path in first integral extends from 0 to ∞ in the half-plane Re, $p > 0$, and in the second and third integral extends from 0 to ∞ in the half-plane $\mathrm{Re}(s) > 0$.

We can find the unknown function $V(s)$ or $V'(s)$ by substituting the asymptotic series (Eq. (10)) in Eq. (14) and taking the inverse Laplace transform which yields

$$V(s) = \sum_{n=1}^{\infty} b_n s^{2n} \quad \text{and} \quad V'(s) = \sum_{n=1}^{\infty} 2nb_n s^{2n-1} \tag{15}$$

where

$$b_n = \frac{a_n}{(2n)!} \approx \frac{(2n+1)(2n-1)}{(2n+2)(2n+4)}(-1)^n \frac{K}{2n} \quad \text{for large } n \tag{16}$$

It is readily established that the series (Eq. (15)) for $V(s)$ and $V'(s)$ converges for $|s| < 1$ and has a singularity at $s = \pm\mathrm{i}$. However, the singularity of $V(s)$ and $V'(s)$ at $s = \pm\mathrm{i}$ and the non-linear term in Eq. (7) would imply that $V(s)$ and $V'(s)$ will also have singularity at $s = \pm 2\mathrm{i}, \pm 3\mathrm{i}, \dots,$ so on. If $y = -\mathrm{i}Y$, $Y \in \mathbb{R}_+$, then the integrand $V(p/y)$ becomes singular at $p = \pm kY$, $k = 1, 2, \dots$. The singularities at $p = +kY$ lie exactly on the integration path in Eq. (14), and therefore, it has to be deformed clockwise to avoid the singularity, as shown in Fig. 1.

Now we study the behavior of $V(s)$ in the neighborhood of the singularity at $s = +k\mathrm{i}$, $k = 1, 2, \dots$. Since $b_n \approx (-1)^n K/2n$ as $n \to \infty$ we see that $V(s)$ behaves like $K \ln(1 + \mathrm{i}(s/k))$ in the neighborhood of the singularity at $s = +k\mathrm{i}$. Therefore, we have

$$V \approx K \ln\left(1 + \mathrm{i}\frac{p}{ky}\right) \quad \text{as} \quad \frac{p}{y} \to +k\mathrm{i} \tag{17}$$

If $y = -\mathrm{i}Y$, $Y \in \mathbb{R}_+$, and $p \to kY_-$, then the value of the above logarithm will be real, and we will have Eq. (17) in the form

$$V \approx K \ln\left(1 - \frac{p}{kY}\right) \quad \text{as} \quad p \to +kY_- \tag{18}$$

But, if $y = -\mathrm{i}Y$, $Y \in \mathbb{R}_+$, and $p \to kY_+$, then the value of the above logarithm will be complex, and since we deform the integration path in clockwise direction near the singularity, we will have Eq. (17) in the form

$$V \approx K\left[\ln\left(\frac{p}{kY} - 1\right) - \mathrm{i}\pi\right] \quad \text{as} \quad p \to kY_+ \tag{19}$$

Therefore, when y is purely imaginary and negative (i.e. $y = -\mathrm{i}Y$, $Y \in \mathbb{R}_+$), the integrand in Eq. (14), in the neighborhood of the singularity at $p = +kY$, $k = 1, 2, \ldots$, is obtained as

$$V\frac{p}{y}\mathrm{e}^{-p} = V\frac{\mathrm{i}p}{Y}\mathrm{e}^{-p} \approx \begin{cases} K\ln(1-(p/kY))\mathrm{e}^{-p} & \text{for} \quad p = kY_- \\ K\ln((p/kY)-1)\mathrm{e}^{-p} - \mathrm{i}\pi K\mathrm{e}^{-p} & \text{for} \quad p = kY_+ \end{cases} \tag{20}$$

Therefore, from Eq. (14), we have $\eta_{\mathrm{i}}(y)$ as

$$\eta_{\mathrm{i}}(y) = \eta_{\mathrm{i}}(-\mathrm{i}Y) \approx \mathrm{PV}\int_0^{\infty} V\frac{\mathrm{i}p}{Y}\mathrm{e}^{-p}\,\mathrm{d}p - \mathrm{i}\pi K\sum_{k=1}^{\infty}\mathrm{e}^{-kY} \tag{21}$$

The integral in Eq. (21) is the Cauchy principal value (PV) integral which excludes the contributions from the singularities at $p = kY$, $k = 1, 2, \ldots$. The leading contribution from the singularities comes from the singularity at $p = Y$, which is equal to $-\mathrm{i}\pi K\mathrm{e}^{-Y}$. For large Y (i.e. $|y| \gg 1$), the Cauchy principal value integral must agree with the asymptotic series (Eq. (10)) with $y = -\mathrm{i}Y$, and hence, we obtain

$$\eta_{\mathrm{i}}(y) = \eta_{\mathrm{i}}(-\mathrm{i}Y) \approx \sum_{n=1}^{\infty}(-1)^n\frac{a_n}{Y^{2n}} - \mathrm{i}\pi K\sum_{k=1}^{\infty}\mathrm{e}^{-kY} \tag{22}$$

It is clear from Eq. (22) that, an exponentially small correction in the inner solution appears in the asymptotic series of the inner solution beyond all orders. Therefore, there should be a corresponding exponentially small correction in the outer solution which will appear in the algebraic asymptotic series of the outer solution beyond all order. When we match the inner solution (Eq. (22)) to the outer solution, we obtain the solution of Eq. (3) as

$$\eta(x) \approx \eta_0(x) + \epsilon^2\left[-10\gamma^2\eta_0(x) + \frac{5}{2}\eta_0^2(x)\right] + \cdots - \frac{\mathrm{i}\pi K}{\epsilon^2}\sum_{k=1}^{\infty}\mathrm{e}^{-k((\pi/2\gamma\epsilon)+\mathrm{i}(x/\epsilon))} \tag{23}$$

When x is purely real, $\eta(x)$ should be real. Therefore, the correct matching will lead to

$$\eta(x) \approx \eta_0(x) + \epsilon^2\left[-10\gamma^2\eta_0(x) + \frac{5}{2}\eta_0^2(x)\right] + \cdots + \frac{\pi K}{\epsilon^2}\sum_{k=1}^{\infty}\mathrm{e}^{-k\pi/2\gamma\epsilon}\sin\frac{k|x|}{\epsilon} \tag{24}$$

Therefore, the symmetric traveling wave solution of the singularly perturbed (sixth-order) Boussinesq Eq. (1) has small amplitude oscillatory behavior at its tail ends which is explicitly given by $\eta(x) \approx (\pi K/\epsilon^2)\sum_{k=1}^{\infty}\mathrm{e}^{-k\pi/2\gamma\epsilon}\sin(k|x|/\epsilon)$ as $x \to \pm\infty$. The dominant term in the above sum is $(\pi K/\epsilon^2)\mathrm{e}^{-\pi/2\gamma\epsilon}\sin(|x|/\epsilon)$. It is worth pointing out that the frequency of oscillation of the oscillatory tails is of $\mathrm{O}(1/\epsilon)$ which is same as predicted in the far-field analysis of Section 2.

4. Perturbation analysis in the Fourier domain

In this section, we will construct the oscillatory tails and estimate their amplitude by transforming the problem into a Fourier domain and using a perturbation analysis in the Fourier domain as in Akylas and Yang [6]. In this method, the amplitude of tail oscillations is determined easily without the need for asymptotic matching in the complex plane, as required in the technique of asymptotics beyond all orders in Section 3. So, taking the Fourier transform of Eq. (4), we obtain

$$\hat{\eta}\frac{1}{\epsilon} f(\tilde{k}) \operatorname{cosech} \frac{\pi \tilde{k}}{2\epsilon\gamma} \tag{25}$$

where

$$f(\tilde{k}) = 3\tilde{k} + \tfrac{15}{2}\tilde{k}^2 + \cdots \quad \text{and} \quad \tilde{k} = k\epsilon \tag{26}$$

In the evaluation of Eq. (25), we have used $\hat{\eta}_0 = 3k \operatorname{cosech}(\pi k/2\gamma)$ and $\tilde{\eta}_0^2 = 3k(k^2+4\gamma^2)\operatorname{cosech}(\pi k/2\gamma)$. Now, taking the Fourier transform of Eq. (3) and substituting Eq. (25) in the resulting transformed equation, we see that, to the leading order in ϵ, $f(\tilde{k})$ satisfies the following Volterra integral equation.

$$\tilde{k}^2(\tilde{k}^2 - 1) f(\tilde{k}) + 2\int_0^{\tilde{k}} f(\tilde{l}) f(\tilde{k} - \tilde{l})\, \mathrm{d}\tilde{l} = 0 \tag{27}$$

It can be easily shown that, the solution of Eq. (27) can be expressed in the form of a power series

$$f(\tilde{k}) = \sum_{m=0}^{\infty} b_m \tilde{k}^{2m+1} \tag{28}$$

where the coefficients b_m satisfy the recurrence relation

$$-\frac{(2m-1)(2m+6)}{(2m+3)(2m+2)} b_m + b_{m-1} + 2\sum_{r=1}^{m-1} \frac{(2m-2r+1)!(2r+1)!}{(2m+3)!} b_r b_{m-r} = 0, \quad m \geq 2 \tag{29}$$

with $b_0 = 3$, $b_1 = 15/2$. As $m \to \infty$, the non-linear term in Eq. (29) becomes less important. So, we obtain $b_m \approx b_{m-1} \approx C$ as $m \to \infty$, where C is a constant. The value of C can be obtained by evaluating the values of b_m from Eq. (29) up to some large values of m. It is found to be $C = 29.96$. So, $C = K/2$, where $K = 59.91$, as obtained in Section 3. Thus, the series (Eq. (28)) for f is seen to be convergent for $|\tilde{k}| < 1$ and have pole singularities at $\tilde{k} = \pm 1$. Therefore

$$f(\tilde{k}) \approx \frac{C\tilde{k}}{1 - \tilde{k}^2} \quad \text{as} \quad \tilde{k} \to \pm 1 \approx -\frac{K}{4(\tilde{k} \mp 1)} \quad \text{as} \quad \tilde{k} \to \pm 1 \tag{30}$$

In view of Eqs. (25) and (30), $\hat{\eta}$ will have pole singularities at $k = \pm 1/\epsilon$ and will be given by

$$\hat{\eta} \approx -\frac{K}{4\epsilon^2(k \mp 1/\epsilon)} \operatorname{cosech} \frac{\pi k}{2\gamma} \quad \text{as} \quad k \to \frac{\pm 1}{\epsilon} \approx \mp \frac{K}{2\epsilon^2(k \mp 1/\epsilon)} \mathrm{e}^{-\pi/2\gamma\epsilon} \quad \text{as} \quad k \to \frac{\pm 1}{\epsilon} \tag{31}$$

Taking the inverse transform of $\hat{\eta}(k)$ given by Eqs. (25) and (31), we obtain

$$\eta(x) = \mathrm{PV}\int_{-\infty}^{\infty} \hat{\eta}(k)\mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k + \int_{C_{-1/\epsilon}} \hat{\eta}(k)\mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k + \int_{C_{1/\epsilon}} \hat{\eta}(k)\mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k \tag{32}$$

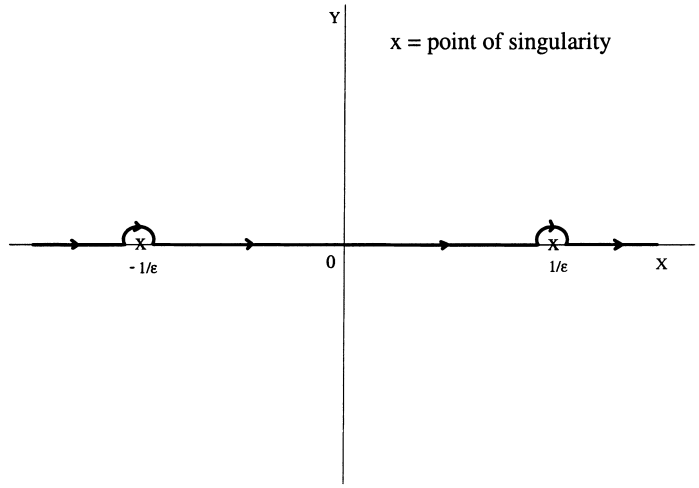

Fig. 2. Deformation of the integration path around the singularities at $k = -1/\epsilon$ and $k = 1/\epsilon$ in a clockwise direction.

where $C_{-1/\epsilon}$ and $C_{1/\epsilon}$ represent the integration path (half circles) near the singularity at $-1/\epsilon$ and $1/\epsilon$, respectively, as shown in Fig. 2. The first integral in Eq. (32) is the Cauchy principal value (PV) integral which must agree with the asymptotic series (Eq. (4)). By the residue theorem, we obtain the second and third integral in Eq. (32) as $-(\mathrm{i}\pi K/2\epsilon^2)\mathrm{e}^{-\pi/2\gamma\epsilon}\mathrm{e}^{-\mathrm{i}x/\epsilon}$ and $(\mathrm{i}\pi K/2\epsilon^2)\mathrm{e}^{-\pi/2\gamma\epsilon}\mathrm{e}^{\mathrm{i}x/\epsilon}$. Therefore, we obtain Eq. (32) in the form

$$\eta(x) \approx \eta_0(x) + \epsilon^2\left[-10\gamma^2\eta_0(x) + \frac{5}{2}\eta_0^2(x)\right] + \cdots + \frac{\pi K}{\epsilon^2}\mathrm{e}^{-\pi/2\gamma\epsilon}\sin\frac{|x|}{\epsilon} \tag{33}$$

Eq. (33) is the required non-local solitary wave solution of the sixth-order (singularly perturbed) Boussinesq Eq. (1). Thus, we have the far-field oscillation in the form $\eta(x) \approx (\pi K/\epsilon^2)\mathrm{e}^{-\pi/2\gamma\epsilon}\sin(|x|/\epsilon)$ as $x \to \pm\infty$. This estimate agrees with the estimate of Section 3 to the leading order.

5. Pseudospectral method

In this section, we solve Eq. (3) numerically using a pseudospectral (collocation) method. The spectral basis functions are chosen suitably a combination of rational Chebychev and radiation basis functions to get the correct solitary wave behavior (Eq. (4)) at the core (near $x = 0$) and oscillatory behavior (Eq. (6)) in the far-field (as $x \to \pm\infty$). Since the method is described in detail in Boyd [10], we only give an outline here.

Since Eq. (3) is non-linear, it is solved iteratively. Suppose $\eta^{(\mathrm{i})}(x)$ is the solution at ith iterate and $\delta\eta^{(\mathrm{i})}(x)$ is a correction to $\eta^{(\mathrm{i})}(x)$ such that $\eta(x) = \eta^{(\mathrm{i})}(x) + \delta\eta^{(\mathrm{i})}(x)$ satisfies Eq. (3). Substituting it in Eq. (3) and linearizing the LHS, we get the following linear inhomogeneous ODE (known as Newton–Kantorovich equation) for the iterative scheme.

$$((1-c^2) + 2\eta^{(\mathrm{i})})\delta\eta^{(\mathrm{i})} + \delta\eta^{(\mathrm{i})}_{xx} + \epsilon^2\delta\eta^{(\mathrm{i})}_{xxxx} = -[((1-c^2) + \eta^{(\mathrm{i})})\eta^{(\mathrm{i})} + \eta^{(\mathrm{i})}_{xx} + \epsilon^2\eta^{(\mathrm{i})}_{xxxx}] \tag{34}$$

This iteration procedure is repeated until the correction $\delta\eta^{(i)}(x)$, or equivalently, the RHS of Eq. (34) becomes negligibly small. The iteration procedure requires an initial guess. For small values, the solitary wave solution (Eq. (4)) is taken as the initial guess. For large values of ϵ, the method of continuation is used to find a suitable initial guess (see [10] for more detail). Now, if we write the solution at ith iterate as

$$\eta^{(i)}(x) = \sum_{n=1}^{N-1} a_n^{(i)} \Phi_n(x) + \Phi_{\mathrm{rad}}(x; A^{(i)}) \tag{35}$$

then the correction to the solution at ith iterate will be given by

$$\delta\eta^{(i)}(x) \approx \sum_{n=1}^{N-1} \delta a_n^{(i)} \Phi_n(x) + \delta A^{(i)} \Phi_{\mathrm{rad},A}(x; A^{(i)}) \tag{36}$$

The amplitude A of the tail oscillations is obtained as a part of the solution along with the spectral coefficients $a_n, n = 1, 2, \dots, N-1$. The spectral basis functions $\Phi_n(x), n = 1, 2, \dots, N-1$ and $\Phi_{\mathrm{rad}}(x; A)$ are constructed as follows (see [10] for more details).

$$\Phi_n(x) = \mathrm{TB}_{2n}(x) - 1 = \cos\left[2n \cot^{-1}\frac{x}{L}\right] - 1, \quad L = \frac{2}{\gamma} \tag{37}$$

and

$$\Phi_{\mathrm{rad}}(x; A) = H(x)\eta_{\mathrm{cn}}(x; A) + H(-x)\eta_{\mathrm{cn}}(-x; A) \tag{38}$$

Since the rational Chebychev functions $\mathrm{TB}_{2n}(x)$ are even and asymptote to 1 as $x \to \pm\infty$, the basis functions $\Phi_n(x)$ are even and decay down to zero at tail ends. Thus, the series $\sum_n^{N-1} a_n \Phi_n(x)$ gives the right behavior of the symmetric core solitary wave with peak at $x = 0$. The oscillatory behavior of the solution at tail ends is visualized by the radiation basis function $\Phi_{\mathrm{rad}}(x; A)$ through its dependence on cnoidal function $\eta_{\mathrm{cn}}(x; A)$ which is given by

$$\eta_{\mathrm{cn}}(x; A) = A \sin\left[\frac{q}{\epsilon}x + \phi\right] + A^2\left[C_1 + C_2 \cos\left[\frac{2q}{\epsilon}x + \phi\right]\right] + A^3 C_3 \sin\left[\frac{3q}{\epsilon}x + \phi\right] + \mathrm{O}(A^4) \tag{39}$$

where $q = q_0 + A^2 q_2 + \mathrm{O}(A^4)$, $q_0 = (1 + 4\epsilon^2\gamma^2)^{1/2}$, $q_2 = \epsilon^4(C_2 - 2C_1)/(2q_0^3 - q_0)$, $C_1 = \epsilon^2/2(q_0^2 - q_0^2)$, $C_2 = \epsilon^2/(30q_0^4 - 6q_0^2)$, $C_3 = \epsilon^4/48(50q_0^8 - 15q_0^4 + q_0^2)$. Thus, the cnoidal function $\eta_{\mathrm{cn}}(x; A)$ agrees with the form of the far-field solution (Eq. (6)) to the leading order in A. Therefore, it describe the far-field behavior more accurately. The phase shift constant $\phi = 0$ corresponds to the case in which both the core solitary wave and the oscillatory tails are in phase. The smoothed step function $H(x)$ is suitably chosen in order to have the asymptotic behavior $H(x) \sim 1$ as $x \to \infty$ and $H(x) \sim 0$ as $x \to -\infty$. For simplicity, we choose

$$H(x) = \tfrac{1}{2}[1 + \tanh(\gamma(x + \phi))] \tag{40}$$

Since we are interested in obtaining symmetric non-local solitary wave solution of Eq. (3) with peak at $x = 0$ and phase shift constant $\phi = 0$, we choose the N spectral grid (collocation) points all on positive real axis given by

$$x_n = L \cot\frac{(2n-1)\pi}{4N}, \quad n = 1, 2, \dots, N \tag{41}$$

At ith iterate, $\eta^{(i)}$, $A^{(i)}$ and $a_n^{(i)}$ are known. We need to compute the corresponding corrections $\delta A^{(i)}$ and $\delta a_n^{(i)}$ from the Newton–Kantorovich Eq. (34). Now, substituting the spectral series (Eq. (36)) into Eq. (34) and demanding that the residual vanish at N collocation points defined above, we obtain the matrix equation $JE = F$, where $E = [\delta a_1^{(i)}, \delta a_2^{(i)}, \dots, \delta a_{N-1}^{(i)}, \delta A^{(i)}]^{\mathrm{T}}$, $F = [F_1^{(i)}, F_2^{(i)}, \dots, F_N^{(i)}]^{\mathrm{T}}$ and $J = [J_{nj}^{(i)}]$ is the Jacobian matrix of the resulting system of equations. Explicitly $J_{nj}^{(i)}$ and $F_n^{(i)}$ for $n = 1, 2, \dots, N$ are expressed as

$$J_{nj}^{(i)} = \begin{cases} [((1-c^2) + 2\eta^{(i)})\phi_j + \phi_{j,xx} + \epsilon^2 \phi_{j,xxxx}]|_{x=x_n} & \text{for} \quad j = 1, 2, \dots, N-1 \\ [((1-c^2) + 2\eta^{(i)})\phi_{\mathrm{rad},A} + \phi_{\mathrm{rad},Axx} + \epsilon^2 \phi_{\mathrm{rad},Axxxx}]|_{x=x_n} & \text{for} \quad j = N \end{cases} \tag{42}$$

and

$$F_n^{(i)} = [((1-c^2) + \eta^{(i)})\eta^{(i)} + \eta_{xx}^{(i)} + \epsilon^2 \eta_{xxxx}^{(i)}]|_{x=x_n} \tag{43}$$

The various derivatives of the basic functions involved in the calculation of Jacobian matrix J through Eq. (42) and RHS column vector F through Eq. (43) can be obtained explicitly. The matrix equation $JE = F$ is solved for E using a direct numerical method such as Gaussian elimination with partial pivoting. The spectral coefficients are corrected through $a_n^{(i+1)} = a_n^{(i)} + \delta a_n^{(i)}$, $n = 1, 2, \dots, N-1$ and $A^{(i+1)} = A^{(i)} + \delta A^{(i)}$. Then the new solution, new Jacobian matrix and new RHS vector are evaluated using the updated values $a_n^{(i+1)}$ and $A^{(i+1)}$. Then the matrix equation is solved again. The iteration procedure is continued until the maximum (L_∞) norm of the vector E, or equivalently, F becomes negligibly small.

6. Numerical results

The analytical estimate of amplitude A of the oscillatory tails for different values of the perturbation parameter ϵ^2 and phase speed c is shown in Table 1. It is observed that, the amplitude A of the oscillatory tails is exponentially small as compared with the amplitude of the core which is approximately equal to $6\gamma^2$ or $1.5(c^2 - 1)$. Also it decreases exponentially fast as the value of ϵ and c decreases.

Table 1
Analytical estimate of the amplitude A of the oscillatory tails for different values of the perturbation parameter ϵ^2 and phase c

ϵ^2	c				
	1.05	1.10	1.15	1.20	1.25
0.0025	0.430405E − 80	0.206454E − 54	0.640770E − 43	0.520739E − 36	0.293631E − 31
0.0100	0.433390E − 38	0.295349E − 25	0.162456E − 19	0.458033E − 16	0.107685E − 13
0.0225	0.301413E − 24	0.106437E − 15	0.704676E − 12	0.138968E − 09	0.523737E − 08
0.0400	0.209484E − 17	0.529839E − 11	0.383407E − 08	0.199358E − 06	0.300032E − 05
0.0625	0.238900E − 13	0.310892E − 08	0.594467E − 06	0.138618E − 04	0.120067E − 03
0.0900	0.112281E − 10	0.201547E − 06	0.158387E − 04	0.216081E − 03	0.129365E − 02
0.1225	0.861493E − 09	0.374824E − 05	0.155859E − 03	0.144863E − 02	0.665788E − 02
0.1600	0.214244E − 07	0.321447E − 04	0.828458E − 03	0.577118E − 02	0.217482E − 01
0.2025	0.252448E − 06	0.165261E − 03	0.293416E − 02	0.163294E − 01	0.527304E − 01
0.2500	0.176862E − 05	0.595707E − 03	0.784714E − 02	0.364899E − 01	0.104153E + 00

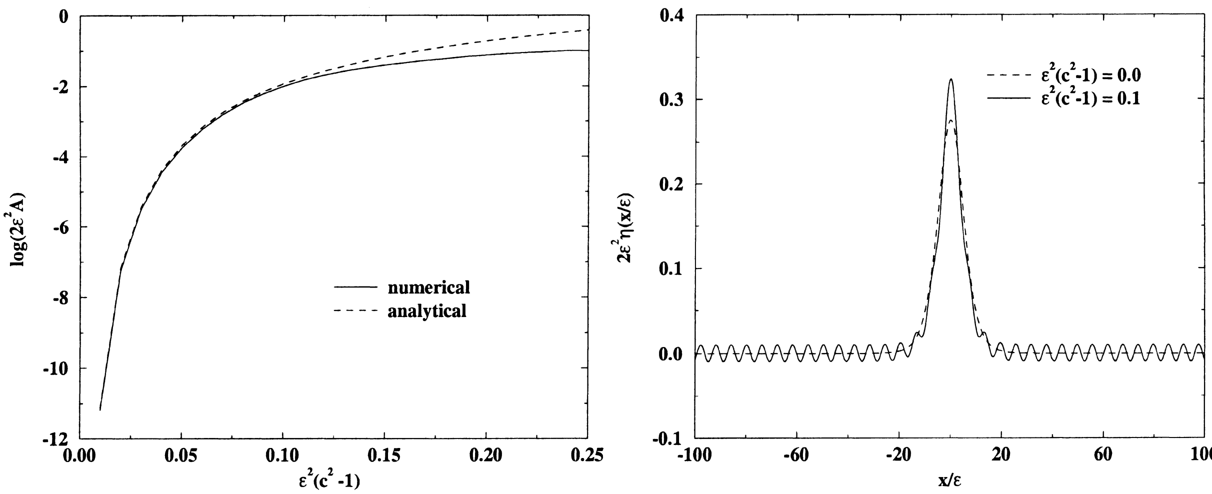

Fig. 3. Left graph: comparison of the numerically computed amplitude (solid lines) of the oscillatory tails with that of the analytical estimate (dashed lines). Right graph: plots for travelling wave solutions of the singularly perturbed (sixth-order) Boussinesq equation for $\epsilon^2(c^2-1)=0.1$ and $\epsilon^2(c^2-1)=0.0$.

The numerical results are obtained for phase shift constant $\phi = 0$ and various values of the perturbation parameter ϵ^2 and phase speed c. However, the results are presented with respect to a combined (group) parameter $\epsilon^2(c^2-1)$. The numerically computed amplitude of the oscillatory tails is compared with the corresponding analytical estimate in the left side graph of Fig. 3. This graph shows the variation of $2\epsilon^2 A$ with the group parameter $\epsilon^2(c^2-1)$. It is observed that the numerically computed amplitude of the far-field oscillations agrees well with the analytical estimate for small values

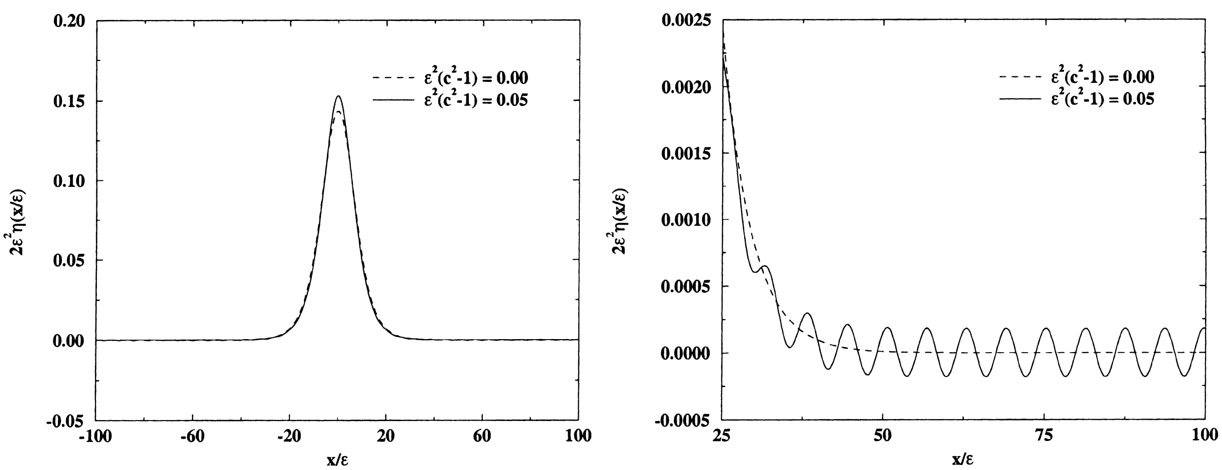

Fig. 4. The plots for traveling wave solutions of the singularly perturbed (sixth-order) Boussinesq equation for $\epsilon^2(c^2-1)=0.05$ and $\epsilon^2(c^2-1)=0.0$. The left side graph of the figure shows the full plot, whereas, the right side graph of figure shows the zoomed plot near the oscillatory tail.

of $\epsilon^2(c^2 - 1)$. However, for larger values of $\epsilon^2(c^2 - 1)$, there is a small discrepancy between the two estimates which is expected since the analytical estimate is based on the asymptotic analysis for $\epsilon \ll 1$. Also, it is to be noted that the amplitude decreases exponentially fast as the value of $\epsilon^2(c^2 - 1)$ decreases.

The right side graph of Fig. 3 shows the numerically computed symmetric weakly non-local solitary wave solution of the sixth-order Boussinesq Eq. (1) for $\epsilon^2(c^2 - 1) = 0.1$. For this moderate value of $\epsilon^2(c^2 - 1)$, the oscillatory tail is clearly visible. However, the oscillatory tail is very (exponentially) small in comparison to the amplitude of the core solitary wave which is centered on the origin $x = 0$. The core in the neighborhood of $x = 0$ is best described by the solution (Eq. (3)). As the value of $\epsilon^2(c^2 - 1)$ decreases, the oscillatory tails decrease and collapse almost into the local solitary wave solution of the fourth-order Boussinesq equation, as seen in the left side graph of Fig. 4. The oscillatory tails are there, but are so small that they are invisible in comparison to the peak of the wave. However, if we zoom near the tail, the oscillations are clearly visible as seen in the right side graph of Fig. 4.

7. Discussions and concluding remarks

In Daripa and Hua [1], a singularly perturbed (sixth-order) Boussinesq equation was introduced as a dispersive regularization of the ill-posed classical (fourth-order) Boussinesq equation. We analyzed this equation to find the traveling wave solutions. On the basis of far-field analyses and heuristic arguments, we established that, unlike the classical solitary waves, the traveling wave solutions of this regularized sixth-order Boussinesq equation cannot vanish in the far-field. Instead, such waves must possess small amplitude fast oscillations at distances far from the core of the waves extending up to infinity. This behavior confirms the numerical prediction of Daripa and Hua [1]. So, the traveling wave solutions of this equation have the behavior of the weakly non-local solitary wave solutions of the singularly perturbed (fifth-order) KdV equation ([5,6,10,14,15]), and the full non-linear water wave equations for $0 < \tau < 1/3$ ([16–19]).

We reviewed various analytical [6,14,15] and numerical [10] methods originally devised to obtain this type of weakly non-local solitary wave solutions of the fifth-order (singular perturbed) KdV equation. Using these methods, we obtain the weakly non-local solitary wave solutions of the regularized sixth-order (singularly perturbed) Boussinesq equations and provide the estimates of the amplitude of oscillations which persist far from the core of the waves. The analytical estimate of the amplitude agreed with that of the numerical estimate for small values of the perturbation parameter. Also, although the analytical estimate of the tail oscillations is similar to that obtained by Akylas and Yang [6], Grimshaw and Joshi [14] and Pomeau et al. [15] for the fifth-order KdV equation, the estimate in the present case is different from their estimates because of the different estimate of the constant K and different relation between the phase speed c of the wave and the core solitary wave width parameter γ.

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