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A fast algorithm for two-dimensional elliptic problems

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In this paper, we extend the work of Daripa et al. [14–16,7] to a larger class of elliptic problems in a variety of domains. In particular, analysis-based fast algorithms to solve inhomogeneous elliptic equations of three different types in three different two-dimensional domains are derived. Dirichlet, Neumann and mixed boundary value problems are treated in all these cases. Three different domains considered are: (i) interior of a circle, (ii) exterior of a circle, and (iii) circular annulus. Three different types of elliptic problems considered are: (i) Poisson equation, (ii) Helmholtz equation (oscillatory case), and (iii) Helmholtz equation (monotone case). These algorithms are derived from an exact formula for the solution of a large class of elliptic equations (where the coefficients of the equation do not depend on the polar angle when written in polar coordinates) based on Fourier series expansion and a one-dimensional ordinary differential equation. The performance of these algorithms is illustrated for several of these problems. Numerical results are presented.

Keywords: Poisson equation, Helmholtz equation, numerical algorithm

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1. Introduction

Modeling of many practical problems in mechanics and other areas of mathematical physics requires solutions of inhomogeneous elliptic equations. Some of the elliptic equations that often arise are the Poisson and the Helmholtz-type equations. Helmholtz-type equations usually appear in scattering theory, acoustics, electromagnetics and time discretization of Navier–Stokes equations, to name just a few. Therefore, availability of fast and accurate algorithms to solve these elliptic equations will allow rapid solution of many practical problems.

There are many numerical approaches to solve elliptic equations such as finite difference, finite element, spectral, wavelet, and integral equation methods. The literature

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on this aspect is too large to survey here. Based on these various approaches, many fast and accurate methods have been constructed in recent years. For example see [1,4,8–11,19,20,23–25,29,30] among many others. The algorithm in the present work is somewhat related to the idea used by Daripa and collaborators in developing fast algorithms for various purposes (see [13–15]).

One of the methods to solve inhomogeneous second order elliptic problems subject to either Dirichlet or Neumann boundary conditions requires the following steps. First, a particular solution is constructed. The difference of the boundary values of this particular solution and given boundary conditions forms the boundary data for an appropriate homogeneous equation. The solution of this homogeneous equation is then calculated using one of the many available fast methods.

The efficiency of an accurate method also depends crucially on the way the particular solution is computed. In Green's function approach, the particular solution can be represented as a multi-dimensional integral whose integrand contains the free space Green's function of the elliptic operator and the inhomogeneous term of the elliptic equation. Analysis based fast and very accurate algorithms to evaluate such integrals arising in solving the nonhomogeneous Cauchy–Riemann equations, the Beltrami equations, and the Poisson equations have been proposed and applied to solve various problems by Daripa and collaborators [5–7,12–16]. The fast algorithm for evaluation of these integrals in a disk is based on the fast Fourier transform (FFT) and recursive relations that make use of only one-dimensional integrals in radial directions. (The use of this algorithm in arbitrary domains has been addressed recently by the authors [3]). This algorithm takes into account the exact contribution of the singularity to the integral and, hence, is also very accurate. Moreover, this algorithm has the asymptotic complexity $O(N \log N)$, where N is the number of unknowns. In actual implementations, these algorithms give solutions in almost $O(N)$ time due to a very low value of the constant (number of operations required per unknown) associated with these algorithms. This is nearly optimal which is very encouraging considering the fact that it is based on classical analysis and various other features some of which are discussed below.

The algorithms presented in this paper are derived through a different formulation for a wider class of elliptic problems. The analysis leading up to these algorithms involves the following steps.

1. Using the FFT, the second order inhomogeneous elliptic equations are reduced to one-dimensional linear ordinary differential equations (ODE) with non-constant coefficients.
2. Appropriate particular solutions of these ODEs are constructed in terms of one-dimensional integrals.
3. Using the particular and the complementary solutions of these ODEs, exact solutions of these ODEs subject to appropriate boundary conditions are formally constructed.
4. Some properties of these one-dimensional integrals are noted including some recursive relations which can reduce computational load significantly.
5. Fast algorithms based on these properties and the FFT are then constructed.

Thus, these algorithms are based on the formal representation of the exact series solutions of elliptic equations, and the numerical error arises mainly due to one-dimensional numerical integration. Since this error can be made, in principle, as small as we please, the method is high-order accurate. The number of operations to achieve a certain accuracy is very low and the complexity (asymptotic operation count) of these algorithms is $O(N \log N)$. The constant hidden behind this order estimate is also very low. Moreover, these algorithms are very simple conceptually, easy to understand, easy to implement, and parallelizable by construction (see [5–7] for parallelization issues of similar algorithms).

Similarity of our approach with other well known approaches such Fourier Analysis Cyclic Reduction (FACR) and other classical methods [21,31,33] is in the use of Fourier series which separates the variables reducing two-dimensional problems to solving independent one-dimensional problems, or equivalently independent algebraic systems. However the similarity stops here. Our approach then differs from others in the way these reduced problems are solved. For example, FACR based methods for the solutions of Poisson's equations for two-dimensional problems on regular grids [21,31,33] use one-dimensional FFTs which decouple the equations giving rise to independent triangular systems. Cyclic reduction and Gaussian elimination (or another set of one-dimensional FFTs and inverse FFTs) are then used to solve the linear systems. In contrast, in our approach the reduced problems are solved by making use of exact analyses and some recursive relations. Since our approach and the simplest version of these FACR methods have the same computational complexity, $O(N \log N)$, it is likely that our method is at least as competitive, but likely to be more accurate for the same computational effort. Our algorithms are also easier to implement, parallelizable by construction, and applicable to a wider class of elliptic problems.

FACR based methods are suitable for Poisson equations, where as our approach here is a unified approach and applies to a host of problems inclding Poisson and two types of Helmholtz equations addressed in this paper. Moreover, the algorithms for all the problems presented are parallelizable by construction and virtually architecture-independent implementions of these algorithms can be done following some of the ideas that we have recently discussed in [6]. At this point, it is worth mentioning that there now exists a host of fast elliptic solvers, in particular Poisson solvers, based on various other principles including the use of FFT, fast multipole method, etc.

The paper is organized as follows. Section 2 is concerned with the formulation of a general class of inhomogeneous elliptic equations in two dimensions in terms of one-dimensional problems by making use of Fourier transforms. The bulk of section 3 describes the method for the Poisson and Helmholtz equations (monotone and oscillatory) and for all three types of domains. All the nine combinations are discussed in detail, even though these do not differ conceptually. However, the detailed discussion is necessary for proper exposition of some subtle points, for proper description of the algorithms, and for ease of implementation of the algorithms discussed in section 4. Section 4 describes the fast algorithm. Numerical results are presented in section 5. Finally, in section 6, we draw conclusions.

2. Inhomogeneous elliptic equations in $\mathbb{R}^2$

We consider elliptic equations in a domain $\Omega \subset \mathbb{R}^2$ which can be an open disc, or an open annulus or the complement of a closed disc, all centered at the origin. The boundary of the domain is denoted by $\partial\Omega$ and the two radii limiting Ω are denoted by $R_1 < R_2$, where R_1 can be zero and R_2 can be infinity.

Let L be an elliptic operator, and let its coefficients have sufficient regularity so that the coefficients of its adjoint L^* are continuous. For a Dirichlet problem associated with the equation

$$Lu = f \quad \text{in } \Omega \tag{2.1}$$

and the boundary condition

$$u = g \quad \text{on } \partial\Omega, \tag{2.2}$$

we assume that f is continuous in $\overline{\Omega}$ and g is continuous on $\partial\Omega$. Therefore the problem has a solution in the classical sense, i.e. $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, satisfies equation (2.1) pointwise and by continuity the boundary conditions on $\partial\Omega$. For exterior problems, the conditions at infinity are also fulfilled.

The data f and h of a Neumann problem with equation (2.1) and boundary condition

$$\partial_n u = h \quad \text{on } \partial\Omega, \tag{2.3}$$

are assumed to be also continuous on $\overline{\Omega}$ and $\partial\Omega$, respectively, and to satisfy appropriate conditions for the existence of classical solutions. Evidently, for the Laplace operator the uniqueness is up to an additive constant and we must have

$$\int_\Omega f \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega} h \, \mathrm{d}t. \tag{2.4}$$

Also, for both exterior problems, Dirichlet and Neumann, we assume that f has a compact support.

We consider now the equation

$$Lu(r,\theta) = f(r,\theta), \quad R_1 < r < R_2,\ 0 \leqslant \theta \leqslant 2\pi, \tag{2.5}$$

where L is written in polar coordinates. We assume in the following that the coefficients of the operator L in terms of polar coordinates are independent of θ, and we write explicitly in the formula the polar coordinates (r,θ) or $r\mathrm{e}^{\mathrm{i}\theta}$ when we use the polar form of L. With the above assumption, for each integer n, there is an ordinary differential operator of second order L_n whose coefficients do not depend on θ satisfying

$$L_n u_n(r) = \mathrm{e}^{-\mathrm{i}n\theta} L\big(u_n(r)\mathrm{e}^{\mathrm{i}n\theta}\big) \tag{2.6}$$

for any $u_n(r) \in C^2(R_1, R_2)$. Writing $L = L_r + a(r)\partial_\theta + b(r)\partial_r\partial_\theta + c(r)\partial_\theta^2$, where the operator L_r depends only on r, and using integration by parts we can verify that L_n satisfies the equation

$$L_n \int_0^{2\pi} u(r,\theta)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta = \int_0^{2\pi} Lu(r,\theta)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta \tag{2.7}$$

for any $u(r,\theta) \in C^2(\Omega)$ and for any integer n and $R_1 < r < R_2$.

Now, for each $r \in (R_1, R_2)$, we write $f(r,\theta)$ and $u(r,\theta)$ as Fourier series on $[0, 2\pi]$,

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)\mathrm{e}^{\mathrm{i}n\theta} \tag{2.8}$$

and

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} u_n(r)\mathrm{e}^{\mathrm{i}n\theta}, \tag{2.9}$$

respectively. Applying equation (2.7) to the solution $u(r,\theta)$ of equation (2.5) and using the above Fourier series expansions we get

$$L_n u_n(r) = f_n(r) \tag{2.10}$$

for any integer number n. Thus, the Fourier coefficients of u satisfy equation (2.10). We state this well-known result as a theorem.

Theorem 2.1. Assume that the coefficients of the operator L in its polar form do not depend on the angle θ. If a solution u of equation (2.5) is written as the Fourier expansion (2.9), then its coefficients $u_n(r)$ are solutions of the equations

$$L_n u_n(r) = f_n(r), \quad R_1 < r < R_2, \tag{2.11}$$

where the operator L_n is given by (2.6) and f_n are the Fourier coefficients of the function f given by (2.8).

Now, let us assume that the boundary data g and h are written as Fourier series

$$g(r,\theta) = \sum_{n=-\infty}^{\infty} g_n(r)\mathrm{e}^{\mathrm{i}n\theta}, \quad (r,\theta) \in \partial\Omega, \tag{2.12}$$

and

$$h(r,\theta) = \sum_{n=-\infty}^{\infty} h_n(r)\mathrm{e}^{\mathrm{i}n\theta}, \quad (r,\theta) \in \partial\Omega, \tag{2.13}$$

respectively. Since the boundaries of the domains are given by either one or two circles, we see that the conditions (2.14) and (2.15) below follow, respectively, from boundary conditions (2.2) and (2.3) for any integer number n.

$$u_n(R_1) = g_n(R_1) \equiv g_n^{(1)}, \qquad u_n(R_2) = g_n(R_2) \equiv g_n^{(2)}, \tag{2.14}$$

and

$$d_r u_n(R_1) = -h_n(R_1) \equiv -h_n^{(1)}, \qquad d_r u_n(R_2) = h_n(R_2) \equiv h_n^{(2)}. \tag{2.15}$$

We have formally included in (2.14) and (2.15) the conditions at the origin and infinity, which the solution must satisfy when written in polar coordinates. Therefore, when Ω is a disc, $u_n(R_1) = g_n(R_1)$ or $d_r u_n(R_1) = h_n(R_1)$ means "$u_n(r)$ has a finite limit when $r \to 0$ for each n". Also, when Ω is the complement of a closed disc, $u_n(R_2) = g_n(R_2)$ or $d_r u_n(R_2) = h_n(R_2)$ means "$u_n(r)$ and/or $d_r u_n(r)$ satisfy appropriate conditions at infinity which arise from the conditions at infinity of the problem in Ω". Thus, the Dirichlet problem defined by (2.1) and (2.2) is reduced to one-dimensional problems given by (2.11) and (2.14), while the Neumann problem given by (2.1) and (2.3) is reduced to one-dimensional problems given by (2.11) and (2.15).

For a point $re^{i\theta}$, which can also be infinity, and the two functions $f(\rho e^{i\tau})$ and $g(\rho e^{i\tau})$, we use the following notation:

$$f(\rho e^{i\tau}) = O(g(\rho e^{i\tau})) \quad \text{as } \rho e^{i\tau} \to re^{i\theta} \quad \text{if } \frac{f(\rho e^{i\tau})}{g(\rho e^{i\tau})} \text{ is bounded} \quad \text{as } \rho e^{i\tau} \to re^{i\theta},$$

$$f(\rho e^{i\tau}) = o(g(\rho e^{i\tau})) \quad \text{as } \rho e^{i\tau} \to re^{i\theta} \quad \text{if } \frac{f(\rho e^{i\tau})}{g(\rho e^{i\tau})} \to 0 \quad \text{as } \rho e^{i\tau} \to re^{i\theta},$$

$$f(\rho e^{i\tau}) \sim g(\rho e^{i\tau}) \quad \text{as } \rho e^{i\tau} \to re^{i\theta} \quad \text{if } \frac{f(\rho e^{i\tau})}{g(\rho e^{i\tau})} \to 1 \quad \text{as } \rho e^{i\tau} \to re^{i\theta}.$$

3. Solution of one-dimensional problems

We look for the solution of one-dimensional problems associated with (2.11), (2.14) and (2.15) in two steps. First, we look for a solution $v_n(r)$ satisfying only equation (2.11),

$$L_n v_n(r) = f_n(r), \quad R_1 < r < R_2, \tag{3.1}$$

of the form

$$v_n(r) = \int_{R_1}^{R_2} f_n(\rho) V_n(\rho, r)\, d\rho, \tag{3.2}$$

where $V_n(\rho, r)$ satisfies in the sense of distributions the equation

$$L_n^* V_n(\rho, r) = \delta(\rho - r), \quad R_1 < \rho < R_2, \tag{3.3}$$

where $\delta(\rho - r)$ is the Dirac delta function. Writing

$$L_n v_n(r) \equiv \alpha_n(r) d_r^2 v_n(r) + \beta_n(r) d_r v_n(r) + \gamma_n(r) v_n(r), \quad R_1 < r < R_2, \tag{3.4}$$

let us assume that the homogeneous equation

$$L_n^* v_n^*(r) \equiv d_r^2\big(\alpha_n(r) v_n^*(r)\big) - d_r\big(\beta_n(r) v_n^*(r)\big) + \gamma_n(r) v_n^*(r) = 0, \quad R_1 < r < R_2, \tag{3.5}$$

has two linearly independent solutions, $v_{n,1}^*(r)$ and $v_{n,2}^*(r)$. In the above, L_n^* is the adjoint of the operator L_n. We seek solutions of equation (3.3) in the form

$$V_n(\rho, r) = \begin{cases} a_n(r) v_{n,1}^*(\rho), & \text{for } R_1 < \rho < r, \\ b_n(r) v_{n,2}^*(\rho), & \text{for } r < \rho < R_2. \end{cases} \tag{3.6}$$

Now the functions $a_n(r)$ and $b_n(r)$ are to be found from the conditions that $V_n(\rho, r)$ is continuous at $\rho = r$,

$$a_n(r) v_{n,1}^*(r) = b_n(r) v_{n,2}^*(r), \tag{3.7}$$

and the jump of its first derivative $\partial_\rho V_n(\rho, r)$ at $\rho = r$ satisfies

$$\alpha_n(r)\big[b_n(r) d_r v_{n,2}^*(r) - a_n(r) d_r v_{n,1}^*(r)\big] = 1, \tag{3.8}$$

where it is assumed that $\alpha_n(r) \neq 0$ for any $R_1 < r < R_2$. From (3.7) and (3.8), we get

$$a_n(r) = \frac{v_{n,2}^*(r)}{\alpha_n(r) D_n^*(r)} \quad \text{and} \quad b_n(r) = \frac{v_{n,1}^*(r)}{\alpha_n(r) D_n^*(r)}, \tag{3.9}$$

where

$$D_n^*(r) = v_{n,1}^*(r) d_r v_{n,2}^*(r) - d_r v_{n,1}^*(r) v_{n,2}^*(r). \tag{3.10}$$

Now, using (3.6) in (3.2) we obtain

$$v_n(r) = a_n(r) \int_{R_1}^{r} f_n(\rho) v_{n,1}^*(\rho)\,\mathrm{d}\rho + b_n(r) \int_{r}^{R_2} f_n(\rho) v_{n,2}^*(\rho)\,\mathrm{d}\rho, \tag{3.11}$$

where $a_n(r)$ and $b_n(r)$ are given by (3.9). We have not yet proven that (3.11) is indeed a solution of (3.1). We do this next.

We notice that pair of equations (3.7)–(3.8) is equivalent to the pair of equations composed of (3.7) and

$$\alpha_n(r)\big[d_r b_n(r) v_{n,2}^*(r) - d_r a_n(r) v_{n,1}^*(r)\big] = -1. \tag{3.12}$$

A simple manipulation of equations (3.7), (3.8) and (3.12) shows that $v_n(r)$ given by (3.11) is a solution of equation (3.1) if $a_n(r)$ and $b_n(r)$ in (3.9) are solutions of the homogeneous form of equation (3.1), evidently under the condition that $\alpha_n(r) d_r D_n^*(r) \neq 0$ for any $R_1 < r < R_2$. To this end, recalling that $v_{n,1}^*(r)$ and $v_{n,2}^*(r)$ are solutions of equation (3.5), we obtain that $D_n^*(r)$ is a solution of the equation

$$\alpha_n(r) d_r D_n^*(r) - \big[\beta_n(r) - 2 d_r \alpha_n(r)\big] D_n^*(r) = 0, \quad R_1 < r < R_2. \tag{3.13}$$

A simple manipulation of this equation shows that $a_n(r)$ and $b_n(r)$ are two linearly independent solutions of the homogeneous form of equation (3.1) if $\alpha_n(r)D_n^*(r) \neq 0$ for any $R_1 < r < R_2$, and $v_{n,1}^*(r)$ and $v_{n,2}^*(r)$ are solutions of equation (3.5). Consequently, we have proved that $v_n(r)$ found in (3.11), i.e.

$$v_n(r) = \frac{v_{n,2}^*(r)}{\alpha_n(r)D_n^*(r)} \int_{R_1}^{r} f_n(\rho)v_{n,1}^*(\rho)\,\mathrm{d}\rho + \frac{v_{n,1}^*(r)}{\alpha_n(r)D_n^*(r)} \int_{r}^{R_2} f_n(\rho)v_{n,2}^*(\rho)\,\mathrm{d}\rho, \quad (3.14)$$

is a solution of equation (3.1).

In the above arguments and analyses, we can interchange $v_{n,1}^*(r)$ and $v_{n,2}^*(r)$ and still arrive at the same proof. Consequently, we can do this interchange in the two integrals in (3.14), but if we want to keep the same formula in (3.10) for $D_n^*(r)$, we have to change the signs in front of these integrals. In this way, we obtain

$$v_n(r) = -\frac{v_{n,1}^*(r)}{\alpha_n(r)D_n^*(r)} \int_{R_1}^{r} f_n(\rho)v_{n,2}^*(\rho)\,\mathrm{d}\rho - \frac{v_{n,2}^*(r)}{\alpha_n(r)D_n^*(r)} \int_{r}^{R_2} f_n(\rho)v_{n,1}^*(\rho)\,\mathrm{d}\rho, \quad (3.15)$$

which is also a solution of equation (3.1).

In the light of what we want to prove next, it is worth recalling from above the following: "$a_n(r)$ and $b_n(r)$ given by (3.9) are two linearly independent solutions of the homogeneous form of equation (3.1), if $\alpha_n(r)D_n^*(r) \neq 0$ for any $R_1 < r < R_2$, and $v_{n,1}^*(r)$ and $v_{n,2}^*(r)$ are two linearly independent solutions of equation (3.5)." Next, we prove the reciprocal statement of it.

Let $v_{n,1}(r)$ and $v_{n,2}(r)$ be two linearly independent solutions of the homogeneous form of equation (3.1) such that $\alpha_n(r)D_n(r) \neq 0$ for any $R_1 < r < R_2$, where

$$D_n(r) = v_{n,1}(r)d_r v_{n,2}(r) - d_r v_{n,1}(r)v_{n,2}(r). \quad (3.16)$$

Replacing a_n and b_n in equations (3.9) by $v_{n,1}$ and $v_{n,2}$ respectively, equations (3.9) become

$$v_{n,1}(r) = \frac{v_{n,2}^*(r)}{\alpha_n(r)D_n^*(r)} \quad \text{and} \quad v_{n,2}(r) = \frac{v_{n,1}^*(r)}{\alpha_n(r)D_n^*(r)}. \quad (3.17)$$

From equations (3.16) and (3.17) we first obtain

$$D_n^*(r) = -\frac{1}{\alpha_n(r)^2 D_n(r)}, \quad (3.18)$$

and then

$$v_{n,1}^*(r) = -\frac{v_{n,2}(r)}{\alpha_n(r)D_n(r)} \quad \text{and} \quad v_{n,2}^*(r) = -\frac{v_{n,1}(r)}{\alpha_n(r)D_n(r)}. \quad (3.19)$$

Now, using the fact that $v_{n,1}(r)$ and $v_{n,2}(r)$ are two linearly independent solutions of the homogeneous form of equation (2.11), we find that $D_n(r)$ is a solution of the equation

$$\alpha_n(r)d_r D_n(r) + \beta_n(r)D_n(r) = 0, \quad R_1 < r < R_2. \quad (3.20)$$

Using this equation, it is easy to prove that $v^*_{n,1}(r)$ and $v^*_{n,2}(r)$ given by (3.19) are solutions of equation (3.5). Concluding, we state these results in the following proposition.

Proposition 3.1. If the coefficients of the operator L_n given by (3.4) satisfy $\alpha_n(r) \neq 0$, $r \in (R_1, R_2)$, $\alpha_n(r) \in C^2(R_1, R_2)$, $\beta_n(r) \in C^1(R_1, R_2)$ and $\gamma_n(r) \in C^0(R_1, R_2)$, then equations (3.17) and (3.19) are reciprocal transformations and establish bijective correspondences between pairs of linearly independent solutions of the homogeneous form of equations (3.1) and (3.5).

Revisiting particular solutions $v_n(r)$ of equation (3.1) given in (3.14) and (3.15), and making use of (3.17) and (3.19), we obtain

$$v_n(r) = -v_{n,1}(r) \int_{R_1}^{r} \frac{v_{n,2}(\rho)}{\alpha_n(\rho) D_n(\rho)} f_n(\rho)\,\mathrm{d}\rho - v_{n,2}(r) \int_{r}^{R_2} \frac{v_{n,1}(\rho)}{\alpha_n(\rho) D_n(\rho)} f_n(\rho)\,\mathrm{d}\rho \tag{3.21}$$

and

$$v_n(r) = v_{n,2}(r) \int_{R_1}^{r} \frac{v_{n,1}(\rho)}{\alpha_n(\rho) D_n(\rho)} f_n(\rho)\,\mathrm{d}\rho + v_{n,1}(r) \int_{r}^{R_2} \frac{v_{n,2}(\rho)}{\alpha_n(\rho) D_n(\rho)} f_n(\rho)\,\mathrm{d}\rho. \tag{3.22}$$

Consequently, we can write:

Theorem 3.1. If the coefficients of the operator L_n given by (3.4) satisfy $\alpha_n(r) \neq 0$, $r \in (R_1, R_2)$, $\alpha_n(r) \in C^2(R_1, R_2)$, $\beta_n(r) \in C^1(R_1, R_2)$ and $\gamma_n(r) \in C^0(R_1, R_2)$, and if $v_{n,1}(r)$ and $v_{n,2}(r)$ are two linearly independent solutions of the homogeneous form of (3.1), then $v_n(r)$ given by (3.21) and (3.22) are solutions of (3.1) subject to the conditions that the integrals in (3.21) and (3.22) are convergent.

Remark 3.1. The above particular solutions can also be derived using the well-known method of variation of parameters, but the formulation leading up to the derivation of the above particular solutions is more general and applicable to more general linear differential operators.

In solving equation (3.1), an appropriate choice of $v_n(r)$ from (3.21) and (3.22) should be made so that the integrals are convergent when the domain Ω is a disc. When Ω is the complement of a closed disc, the Fourier coefficients $f_n(r)$ have compact support because f has compact support by assumption. Therefore, in this case, the integrals in (3.21) and (3.22) are always convergent and hence, either of these equations can be chosen as v_n. When Ω is an annulus, either (3.21) or (3.22) can be chosen as a particular solution of (3.1).

In this paper, we illustrate the proposed method for three types of operators: the Laplace and the two Helmholtz operators (monotonic and oscillatory) all of which satisfy the constraints of theorem 2.1.

Poisson equation. In this case, we have in rectangular coordinates the Laplace operator $Lu \equiv \Delta u$, and $Lu(r,\theta) \equiv (1/r)\partial_r(r\partial_r u) + (1/r^2)\partial_\theta^2 u$ in polar coordinates. Using (2.6), equation (3.1) is written as

$$L_n v_n(r) \equiv \frac{1}{r} d_r\big(r d_r v_n(r)\big) - \frac{n^2}{r^2} v_n(r) = f_n(r), \quad R_1 < r < R_2. \tag{3.23}$$

We have $\alpha_n(r) = 1$, and we can choose

$$\begin{aligned} &v_{0,1}(r) = 1, && v_{0,2}(r) = \log(r), \\ &v_{n,1}(r) = r^{|n|}, && v_{n,2}(r) = r^{-|n|} \quad \text{for } n \neq 0. \end{aligned} \tag{3.24}$$

From (3.16) we get

$$D_0(r) = \frac{1}{r}, \quad \text{and} \quad D_n(r) = -\frac{2|n|}{r} \quad \text{for } n \neq 0.$$

Applying (3.21) we obtain

$$v_n(r) = \frac{1}{2|n|}\int_{R_1}^{r} \rho\left(\frac{r}{\rho}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho + \frac{1}{2|n|}\int_{r}^{R_2} \rho\left(\frac{\rho}{r}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho \quad \text{for } n \neq 0, \tag{3.25}$$

and

$$v_0(r) = -\int_{R_1}^{r} \rho\log(\rho) f_0(\rho)\,\mathrm{d}\rho - \int_{r}^{R_2} \rho\log(r) f_0(\rho)\,\mathrm{d}\rho. \tag{3.26}$$

Also, from (3.22) we get

$$v_n(r) = -\frac{1}{2|n|}\int_{R_1}^{r} \rho\left(\frac{\rho}{r}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho - \frac{1}{2|n|}\int_{r}^{R_2} \rho\left(\frac{r}{\rho}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho \quad \text{for } n \neq 0, \tag{3.27}$$

and

$$v_0(r) = \int_{R_1}^{r} \rho\log(r) f_0(\rho)\,\mathrm{d}\rho + \int_{r}^{R_2} \rho\log(\rho) f_0(\rho)\,\mathrm{d}\rho, \tag{3.28}$$

under the condition that the integrals are convergent. Now, we make the following remarks concerning the convergence of the integrals.

Remark 3.2. Except for the case $R_1 = 0$ and $|n| > 1$, where the first integral in (3.25) can be divergent (depending on f_n), the integrals in (3.25)–(3.28) are convergent.

We remind that the function f has compact support and is bounded on $\overline{\Omega}$. Therefore, its Fourier coefficients f_n have compact support and are bounded on $[R_1, R_2]$ for any integer n.

Helmholtz equation (oscillatory case). The operator for this equation is $Lu \equiv \Delta u + k^2 u$. Using the polar coordinates we have $Lu(r,\theta) \equiv (1/r)\partial_r(r\partial_r u) + (1/r^2)\partial_\theta^2 u + k^2 u$, and from (2.6) we get equation (3.1) as

$$L_n v_n(r) \equiv \frac{1}{r} d_r\big(r d_r v_n(r)\big) + \frac{k^2 r^2 - n^2}{r^2} v_n(r) = f_n(r), \quad R_1 < r < R_2, \tag{3.29}$$

which is a Bessel's differential equation. Here we have again $\alpha_n(r) = 1$ (see (3.4)). The two linearly independent solutions of homogeneous form of equation (3.29) are chosen as

$$v_{n,1}(r) = J_n(kr) \quad \text{and} \quad v_{n,2}(r) = Y_n(kr) \quad \text{for any } n, \tag{3.30}$$

where $J_n(r)$ and $Y_n(r)$ are Bessel functions of the first and the second kind of order n, respectively. Since

$$J_{-n}(r) = (-1)^n J_n(r), \quad \text{and} \quad Y_{-n}(r) = (-1)^n Y_n(r), \quad n = 0, 1, 2, \ldots, \tag{3.31}$$

they satisfy

$$J_{-n}(kr) = (-1)^n J_n(kr), \quad \text{and} \quad Y_{-n}(kr) = (-1)^n Y_n(kr). \tag{3.32}$$

Also, we have

$$r\big[J_n(r) d_r Y_n(r) - Y_n(r) d_r J_n(r)\big] = \frac{2}{\pi} \quad \text{for any } n, \tag{3.33}$$

and consequently, from (3.16),

$$D_n(r) = \frac{2}{\pi r} \quad \text{for any } n \text{ and } k.$$

Now, from (3.21) and (3.22), we get

$$v_n(r) = -\frac{\pi}{2}\int_{R_1}^{r} \rho J_n(kr) Y_n(k\rho) f_n(\rho)\,\mathrm{d}\rho - \frac{\pi}{2}\int_{r}^{R_2} \rho Y_n(kr) J_n(k\rho) f_n(\rho)\,\mathrm{d}\rho, \tag{3.34}$$

and

$$v_n(r) = \frac{\pi}{2}\int_{R_1}^{r} \rho Y_n(kr) J_n(k\rho) f_n(\rho)\,\mathrm{d}\rho + \frac{\pi}{2}\int_{r}^{R_2} \rho J_n(kr) Y_n(k\rho) f_n(\rho)\,\mathrm{d}\rho. \tag{3.35}$$

For the purposes below, we also need the following asymptotic behavior as $r \to 0$:

$$\begin{aligned} &J_n(r) \sim \frac{r^n}{2^n n!} \quad \text{and} \quad J_{-n}(r) \sim \frac{(-r)^n}{2^n n!}, \quad n \geqslant 0, \\ &Y_0(r) \sim \frac{2}{\pi}\log\left(\frac{r}{2}\right), \qquad Y_n(r) \sim \frac{-(n-1)!}{\pi}\left(\frac{2}{r}\right)^n, \quad \text{and} \\ &Y_n(r) \sim \frac{-(n-1)!}{\pi}\left(\frac{-2}{r}\right)^n, \quad n > 0. \end{aligned} \tag{3.36}$$

Therefore, we have:

Remark 3.3. Except for the case $R_1 = 0$ and $|n| > 1$, when the first integral in (3.34) can be divergent (depending on f_n), the integrals in (3.34)–(3.35) are convergent.

In section 3.2.2 we will also need the following linearly independent solutions for this operator, which can be obtained using the Bessel functions of the third kind of order n, also known as Hankel functions of order n.

$$H_n^{(1)}(r) = J_n(r) + \mathrm{i}Y_n(r) \quad \text{and} \quad H_n^{(2)}(r) = J_n(r) - \mathrm{i}Y_n(r). \tag{3.37}$$

Since

$$H_n^{(1)}(kr) = J_n(kr) + \mathrm{i}Y_n(kr) \quad \text{and} \quad H_n^{(2)}(kr) = J_n(kr) - \mathrm{i}Y_n(kr) \tag{3.38}$$

are also linearly independent solutions of the homogeneous form of the Bessel differential equation (3.29), we can take

$$v_{n,1}(r) = H_n^{(1)}(kr) \quad \text{and} \quad v_{n,2}(r) = H_n^{(2)}(kr) \quad \text{for any } n. \tag{3.39}$$

Now, it follows from (3.16), (3.33) and the above definition of the Hankel functions that

$$D_n(r) = \frac{-4\mathrm{i}}{\pi r} \quad \text{for any } n \text{ and } k.$$

Again, from (3.21) and (3.22), we get

$$v_n(r) = -\frac{\mathrm{i}\pi}{4}\int_{R_1}^{r} \rho H_n^{(1)}(kr)H_n^{(2)}(k\rho)f_n(\rho)\,\mathrm{d}\rho - \frac{\mathrm{i}\pi}{4}\int_{r}^{R_2} \rho H_n^{(2)}(kr)H_n^{(1)}(k\rho)f_n(\rho)\,\mathrm{d}\rho \tag{3.40}$$

and

$$v_n(r) = \frac{\mathrm{i}\pi}{4}\int_{R_1}^{r} \rho H_n^{(2)}(kr)H_n^{(1)}(k\rho)f_n(\rho)\,\mathrm{d}\rho + \frac{\mathrm{i}\pi}{4}\int_{r}^{R_2} \rho H_n^{(1)}(kr)H_n^{(2)}(k\rho)f_n(\rho)\,\mathrm{d}\rho. \tag{3.41}$$

From (3.31) and (3.37) we have

$$H_{-n}^{(1)}(r) = (-1)^n H_n^{(1)}(r) \quad \text{and} \quad H_{-n}^{(2)}(r) = (-1)^n H_n^{(2)}(r), \tag{3.42}$$

and then,

$$H_{-n}^{(1)}(kr) = (-1)^n H_n^{(1)}(kr) \quad \text{and} \quad H_{-n}^{(2)}(kr) = (-1)^n H_n^{(2)}(kr). \tag{3.43}$$

Using again (3.36) we have:

Remark 3.4. Except for the case $R_1 = 0$ and $|n| > 1$, when the first integrals in (3.40) and (3.41) can be divergent (depending on f_n), the integrals in (3.40)–(3.41) are convergent.

Helmholtz equation (monotonic case). The operator for this equation is $Lu \equiv \Delta u - k^2 u$. As above this operator is written in polar coordinates as $Lu(r,\theta) \equiv (1/r)\partial_r(r\partial_r u) + (1/r^2)\partial_\theta^2 u - k^2 u$, and from (2.6) we get equation (3.1) as

$$L_n v_n(r) \equiv \frac{1}{r} d_r\big(r d_r v_n(r)\big) - \frac{k^2 r^2 + n^2}{r^2} v_n(r) = f_n(r), \quad R_1 < r < R_2, \tag{3.44}$$

which is a modified Bessel's differential equation. Also, $\alpha_n(r) = 1$, and we take

$$v_{n,1}(r) = I_n(kr) \quad \text{and} \quad v_{n,2}(r) = K_n(kr) \quad \text{for any } n, \tag{3.45}$$

where $I_n(r)$ and $K_n(r)$ are modified Bessel functions of the first and the second kind of order n respectively. Since for all n,

$$I_n(r) = \mathrm{i}^{-n} J_n(\mathrm{i}r) \quad \text{and} \quad K_n(r) = \frac{-\pi}{2}\,\mathrm{i}^n\big[Y_n(\mathrm{i}r) - \mathrm{i}J_n(\mathrm{i}r)\big], \tag{3.46}$$

we have

$$I_n(kr) = \mathrm{i}^{-n} J_n(\mathrm{i}kr) \quad \text{and} \quad K_n(kr) = \frac{-\pi}{2}\,\mathrm{i}^n\big[Y_n(\mathrm{i}kr) - \mathrm{i}J_n(\mathrm{i}kr)\big]. \tag{3.47}$$

Using (3.45) for $v_{n,1}(r)$ and $v_{n,2}(r)$ in (3.16), and making use of (3.33) and (3.47), we obtain after some manipulation

$$D_n(r) = -\frac{1}{r}.$$

Therefore, from (3.21) and (3.22) we have

$$v_n(r) = \int_{R_1}^{r} \rho I_n(kr) K_n(k\rho) f_n(\rho)\,\mathrm{d}\rho + \int_{r}^{R_2} \rho K_n(kr) I_n(k\rho) f_n(\rho)\,\mathrm{d}\rho \tag{3.48}$$

and

$$v_n(r) = -\int_{R_1}^{r} \rho K_n(kr) I_n(k\rho) f_n(\rho)\,\mathrm{d}\rho - \int_{r}^{R_2} \rho I_n(kr) K_n(k\rho) f_n(\rho)\,\mathrm{d}\rho. \tag{3.49}$$

It is well known (also follows from (3.31) and (3.46)) that

$$I_{-n}(r) = I_n(r) \quad \text{and} \quad K_{-n}(r) = K_n(r), \quad n = 0, 1, 2, \ldots, \tag{3.50}$$

and therefore,

$$I_{-n}(kr) = I_n(kr) \quad \text{and} \quad K_{-n}(kr) = K_n(kr). \tag{3.51}$$

For the purposes below, we also need the following asymptotic behaviors (which also follows from (3.36) and (3.46)) of these functions as $r \to 0$.

$$\begin{aligned} &I_n(r) \sim \frac{r^{|n|}}{2^{|n|}|n|!}, \quad \text{for any } n, \\ &K_0(r) \sim -\log\left(\frac{r}{2}\right), \quad \text{and} \quad K_n(r) \sim \frac{(|n|-1)!}{2}\left(\frac{2}{r}\right)^{|n|}, \quad \text{for } n \neq 0. \end{aligned} \tag{3.52}$$

Therefore, regarding the convergence of the integrals in (3.48) and (3.49) we can conclude that.

Remark 3.5. Except for the case $R_1 = 0$ and $|n| > 1$, when the first integral in (3.48) can be divergent (depending on f_n), the integrals in (3.48)–(3.49) are convergent.

The second step in finding the solution $u_n(r)$ of the problems associated with equation (2.11) is to write it as a linear combination of $v_{n,1}(r)$, $v_{n,2}(r)$ and one of the two forms of $v_n(r)$ given by (3.21) and (3.22), with convergent integrals such that appropriate boundary conditions ((2.14) or (2.15) or mixed) are satisfied. We do that separately for each of the three types of domains.

3.1. *Interior circular domains*

In this case $R_1 = 0$, and $R_2 = R$, where R is the radius of the disc. Since one of the linearly independent solutions diverges, we look for a solution of equation (3.1) of the form

$$u_n(r) = v_n(r) + c_n(R)w_n(r), \quad 0 < r < R. \tag{3.53}$$

Here $v_n(r)$ is given by either (3.21) or (3.22) as discussed previously, and $w_n(r)$ is one of the two linearly independent solutions of the homogeneous form of equation (3.1), $v_{n,1}$ or $v_{n,2}$, that is bounded at the origin. Therefore, to have (2.14), we can calculate $c_n(R)$ from the equation

$$c_n(R)w_n(R) = g_n - v_n(R) \tag{3.54}$$

for a Dirichlet problem. Also, to have (2.15) we get $c_n(R)$ from the equation

$$c_n(R)d_r w_n(R) = h_n - d_r v_n(R) \tag{3.55}$$

for a Neumann problem. Then we have:

Proposition 3.2. Under the conditions of theorem 2.1, the Fourier coefficients $u_n(r)$ of the solution $u(r, \theta)$ of (2.1) inside a disc of radius R with its center at the origin are given by

$$u_n(r) = v_n(r) + \frac{g_n - v_n(R)}{w_n(R)} w_n(r), \quad 0 < r < R, \text{ if } w_n(R) \neq 0, \tag{3.56}$$

for the Dirichlet problem (2.1), (2.2), and by

$$u_n(r) = v_n(r) + \frac{h_n - d_r v_n(R)}{d_r w_n(R)} w_n(r), \quad 0 < r < R, \text{ if } d_r w_n(R) \neq 0, \tag{3.57}$$

for the Neumann problem (2.1), (2.3), where $w_n(r)$ is one of the two linearly independent solutions of the homogeneous form of equation (3.1) satisfying the conditions at zero, and $v_n(r)$ is one of (3.21) and (3.22) also satisfying the same conditions at zero.

Below, we provide explicit expression for the solution $u_n(r)$ associated with each of the three types of equations mentioned in the previous section all of which satisfy the constraints of theorem 2.1.

3.1.1. Poisson equation

As $r \to 0$, the second integrals in (3.25) and (3.26) tend to infinity as $r \to 0$, while $v_n(r)$ and $v_0(r)$ given, respectively, in (3.27) and (3.28), namely

$$v_0(r) = \int_0^r \rho \log(r) f_0(\rho)\, \mathrm{d}\rho + \int_r^R \rho \log(\rho) f_0(\rho)\, \mathrm{d}\rho,$$
$$v_n(r) = -\frac{1}{2|n|} \int_0^r \rho \left(\frac{\rho}{r}\right)^{|n|} f_n(\rho)\, \mathrm{d}\rho - \frac{1}{2|n|} \int_r^R \rho \left(\frac{r}{\rho}\right)^{|n|} f_n(\rho)\, \mathrm{d}\rho \quad \text{for } n \neq 0, \tag{3.58}$$

are bounded. Also, $v_{0,2}(r)$ and $v_{n,2}(r)$ given by (3.24) tend to infinity as $r \to 0$, while

$$w_0(r) = 1 \quad \text{and} \quad w_n(r) = r^{|n|}$$

are bounded. Consequently, the above functions satisfy the conditions at zero as required by proposition 3.2. We see that $w_0(r)$, $w_n(r)$ and $d_r w_n(r)$, $n \neq 0$, are different from zero for any $r > 0$, and therefore we can calculate the corresponding $c_n(R)$ in (3.54) and (3.55). For the Neumann problem with $n = 0$, (3.55) gives

$$c_0(R) \cdot 0 = h_0 - \frac{1}{R} \int_0^R \rho f_0(\rho)\, \mathrm{d}\rho.$$

But (2.4) can be written as $Rh_0 = \int_0^R \rho f_0(\rho)\, \mathrm{d}\rho$, i.e. the right-hand side in the above equation is also zero. Therefore, as we have already known, c_0 is an arbitrary constant. Consequently, we get the following from proposition 3.2.

Corollary 3.1. The Fourier coefficients $u_n(r)$ of the solution $u(r, \theta)$ of the Poisson equation inside a disc of radius R with its center at the origin are given by

$$u_0(r) = v_0(r) + g_0 - \int_0^R \rho \log(R) f_0(\rho)\, \mathrm{d}\rho,$$
$$u_n(r) = v_n(r) + \left(\frac{r}{R}\right)^{|n|} \left[g_n + \frac{1}{2|n|} \int_0^R \rho \left(\frac{\rho}{R}\right)^{|n|} f_n(\rho)\, \mathrm{d}\rho \right] \quad \text{for } n \neq 0 \tag{3.59}$$

for the Dirichlet problem (2.1), (2.2), and as

$$u_0(r) = v_0(r) + c_0,$$
$$u_n(r) = v_n(r) + \left(\frac{r}{R}\right)^{|n|} \left[\frac{R}{|n|} h_n - \frac{1}{2|n|} \int_0^R \rho \left(\frac{\rho}{R}\right)^{|n|} f_n(\rho)\, \mathrm{d}\rho \right] \quad \text{for } n \neq 0 \tag{3.60}$$

for the Neumann problem (2.1), (2.3), where c_0 is an arbitrary real constant, and $v_0(r)$ and $v_n(r)$ are given by (3.58).

3.1.2. Helmholtz equation (oscillatory case)

The operator for this equation is $Lu \equiv \Delta u + k^2 u$. Taking into account (3.36), the second integral in (3.34) tends to infinity when r approaches zero, while $v_n(r)$ given by (3.35),

$$v_n(r) = \frac{\pi}{2}\int_0^r \rho Y_n(kr)J_n(k\rho)f_n(\rho)\,\mathrm{d}\rho + \frac{\pi}{2}\int_r^R \rho J_n(kr)Y_n(k\rho)f_n(\rho)\,\mathrm{d}\rho, \tag{3.61}$$

has a finite limit. Also, $v_{n,2}(r)$ in (3.30) tends to infinity as $r \to 0$, and

$$w_n(r) = J_n(kr)$$

has a finite limit. Consequently, we use the above mentioned $v_n(r)$ and $w_n(r)$ in the proposition 3.2. Concerning the derivatives in (3.57) for the Neumann problem, we use

$$d_r J_n(r) = \frac{1}{2}\big[J_{n-1}(r) - J_{n+1}(r)\big] \quad \text{and} \quad d_r Y_n(r) = \frac{1}{2}\big[Y_{n-1}(r) - Y_{n+1}(r)\big], \tag{3.62}$$

obtaining

$$d_r J_n(kr) = \frac{k}{2}\big[J_{n-1}(kr) - J_{n+1}(kr)\big] \quad \text{and} \quad d_r Y_n(kr) = \frac{k}{2}\big[Y_{n-1}(kr) - Y_{n+1}(kr)\big]. \tag{3.63}$$

Now, from proposition 3.2 we get:

Corollary 3.2. The Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ of the Helmholtz equation $Lu \equiv \Delta u + k^2 u = f$ inside a disc of radius R with its center at the origin are given by

$$u_n(r) = v_n(r) + \frac{J_n(kr)}{J_n(kR)}\left[g_n - \frac{\pi}{2}\int_0^R \rho Y_n(kR)J_n(k\rho)f_n(\rho)\,\mathrm{d}\rho\right], \tag{3.64}$$

if $J_n(kR) \neq 0$ for any n, for the Dirichlet problem (2.1), (2.2), and as

$$\begin{aligned} u_n(r) = {} & v_n(r) + \frac{J_n(kr)}{J_{n-1}(kR) - J_{n+1}(kR)} \\ & \times \left\{\frac{2}{k}h_n - \frac{\pi}{2}\int_0^R \rho\big[Y_{n-1}(kR) - Y_{n+1}(kR)\big]J_n(k\rho)f_n(\rho)\,\mathrm{d}\rho\right\}, \end{aligned} \tag{3.65}$$

if $J_{n-1}(kR) \neq J_{n+1}(kR)$ for any n, for the Neumann problem (2.1), (2.3), where $v_n(r)$ is given by (3.61).

3.1.3. Helmholtz equation (monotone case)

The operator for this equation is $Lu \equiv \Delta u - k^2 u$. Similar to the other Helmholtz equation, taking into account (3.36) and (3.46) we find that we have to choose $v_n(r)$ in (3.49)

$$v_n(r) = -\int_0^r \rho K_n(kr)I_n(k\rho)f_n(\rho)\,\mathrm{d}\rho - \int_r^R \rho I_n(kr)K_n(k\rho)f_n(\rho)\,\mathrm{d}\rho, \tag{3.66}$$

and $v_{n,1}(r)$ in (3.45)

$$w_n(r) = I_n(kr)$$

(because they satisfy the conditions at zero) in the proposition 3.2. From (3.46) and (3.62) we get

$$d_r I_n(r) = \frac{1}{2}\big[I_{n-1}(r) + I_{n+1}(r)\big] \quad \text{and} \quad d_r K_n(r) = \frac{-1}{2}\big[K_{n-1}(r) + K_{n+1}(r)\big], \tag{3.67}$$

and therefore,

$$d_r I_n(kr) = \frac{k}{2}\big[I_{n-1}(kr) + I_{n+1}(kr)\big] \quad \text{and} \quad d_r K_n(kr) = \frac{-k}{2}\big[K_{n-1}(kr) + K_{n+1}(kr)\big]. \tag{3.68}$$

In this case, the following corollary follows from proposition 3.2.

Corollary 3.3. The Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ of the Helmholtz equation $Lu \equiv \Delta u - k^2 u = f$ inside a disc of radius R with its center at the origin are given by

$$u_n(r) = v_n(r) + \frac{I_n(kr)}{I_n(kR)}\left[g_n + \int_0^R \rho K_n(kR) I_n(k\rho) f_n(\rho)\,\mathrm{d}\rho\right], \tag{3.69}$$

if $I_n(kR) \neq 0$ for any n, for the Dirichlet problem (2.1), (2.2), and as

$$u_n(r) = v_n(r) + \frac{I_n(kr)}{I_{n-1}(kR) + I_{n+1}(kR)} \times \left\{\frac{2}{k}h_n - \int_0^R \rho\big[K_{n-1}(kR) + K_{n+1}(kR)\big] I_n(k\rho) f_n(\rho)\,\mathrm{d}\rho\right\}, \tag{3.70}$$

if $I_{n-1}(kR) + I_{n+1}(kR) \neq 0$ for any n, for the Neumann problem (2.1), (2.3), where $v_n(r)$ is given by (3.66).

3.2. Exterior circular domains

The domain now is the exterior of a closed disc of radius $R = R_1 \neq 0$ and, using our notation, we have $R_2 = \infty$. Consequently, we look for a solution of equation (3.1) of the form

$$u_n(r) = v_n(r) + c_n(R) w_n(r), \quad R < r < \infty, \tag{3.71}$$

and we assume that $v_n(r)$ (one of those written in (3.21) or (3.22)) as well as $w_n(r)$ (one of the two linearly independent solutions of the homogeneous form of equation (3.1)) satisfy the conditions at infinity arising from the conditions at infinity of the problem in Ω. Now, $u_n(r)$ in (3.71) will satisfy the boundary conditions (2.14) or (2.15) if we can find $c_n(R)$ such that

$$c_n(R) w_n(R) = g_n - v_n(R), \tag{3.72}$$

for a Dirichlet problem, or

$$c_n(R)d_r w_n(R) = -h_n - d_r v_n(R), \tag{3.73}$$

for a Neumann problem, respectively. Similar to proposition 3.2, we have:

Proposition 3.3. Under the conditions of theorem 2.1, the Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ of (2.1) in the exterior of a closed disc of radius R with its center at the origin are given by

$$u_n(r) = v_n(r) + \frac{g_n - v_n(R)}{w_n(R)} w_n(r), \quad R < r < \infty, \text{ if } w_n(R) \neq 0, \tag{3.74}$$

for the Dirichlet problem (2.1), (2.2), and by

$$u_n(r) = v_n(r) - \frac{h_n + d_r v_n(R)}{d_r w_n(R)} w_n(r), \quad R < r < \infty, \text{ if } d_r w_n(R) \neq 0, \tag{3.75}$$

for the Neumann problem (2.1), (2.3), where $w_n(r)$ is one of the two linearly independent solutions of the homogeneous form of equation (3.1) satisfying the conditions to infinity, and $v_n(r)$ is one of (3.21) and (3.22) also satisfying the same conditions at infinity.

Below, we provide explicit expression for the solution $u_n(r)$ associated with each of the three types of equations in the exterion domain.

3.2.1. Poisson equation

By definition, the solution of the exterior problem in the domain Ω satisfies the conditions at infinity if it is bounded at infinity. Consequently, the particular and the complementary (i.e. $v_n(r)$ from (3.25)–(3.28) and $w_n(r)$ from (3.24), respectively) solutions of equation (3.23) bounded at infinity are chosen. We see that the first integral in (3.28) and (3.25) tends to infinity as r approaches infinity, while $v_0(r)$ and $v_n(r)$ given respectively in (3.26) and (3.27),

$$\begin{aligned} v_0(r) &= -\int_R^r \rho \log(\rho) f_0(\rho)\,\mathrm{d}\rho - \int_r^\infty \rho \log(r) f_0(\rho)\,\mathrm{d}\rho, \\ v_n(r) &= -\frac{1}{2|n|}\int_R^r \rho\left(\frac{\rho}{r}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho - \frac{1}{2|n|}\int_r^\infty \rho\left(\frac{r}{\rho}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho \quad \text{for } n \neq 0, \end{aligned} \tag{3.76}$$

have a finite limit. In (3.24), $v_{0,2}(r)$ and $v_{n,1}(r)$ tend to infinity as $r \to \infty$, but the other two, written in the notation used in proposition 3.3 as

$$w_0(r) = 1 \quad \text{and} \quad w_n(r) = r^{-|n|}$$

have a finite limit at infinity. Therefore the above written functions satisfy the conditions at infinity as required by proposition 3.3. As in the case of the interior circular domains, the solution for the Neumann problem is found up to an additive constant c_0. Therefore, from proposition 3.3 we get the following:

Corollary 3.4. The Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ of the Poisson equation in the exterior of a closed disc of radius R with its center at the origin are given by

$$\begin{aligned} u_0(r) &= v_0(r) + g_0 + \int_R^\infty \rho \log(R) f_0(\rho)\,\mathrm{d}\rho, \\ u_n(r) &= v_n(r) + \left(\frac{R}{r}\right)^{|n|}\left[g_n + \frac{1}{2|n|}\int_R^\infty \rho\left(\frac{R}{\rho}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho\right] \quad \text{for } n \neq 0 \end{aligned} \tag{3.77}$$

for the Dirichlet problem (2.1), (2.2), and as

$$\begin{aligned} u_0(r) &= v_0(r) + c_0, \\ u_n(r) &= v_n(r) + \left(\frac{R}{r}\right)^{|n|}\left[\frac{R}{|n|}h_n - \frac{1}{2|n|}\int_R^\infty \rho\left(\frac{R}{\rho}\right)^{|n|} f_n(\rho)\,\mathrm{d}\rho\right] \quad \text{for } n \neq 0 \end{aligned} \tag{3.78}$$

for the Neumann problem (2.1), (2.3), where c_0 is an arbitrary real constant, and $v_0(r)$ and $v_n(r)$ are given by (3.76).

3.2.2. Helmholtz equation (oscillatory case)

The operator for this equation is $Lu \equiv \Delta u + k^2 u$. For this operator, the conditions at infinity of the problems (2.1), (2.2), and (2.1), (2.3) are given by the Sommerfeld radiation condition (see, e.g., [17])

$$u\big(r\mathrm{e}^{\mathrm{i}\theta}\big) = \mathrm{O}\big(1/\sqrt{r}\big) \quad \text{and} \quad \partial_r u\big(r\mathrm{e}^{\mathrm{i}\theta}\big) - \mathrm{i}ku\big(r\mathrm{e}^{\mathrm{i}\theta}\big) = \mathrm{o}\big(1/\sqrt{r}\big) \quad \text{as } r \to \infty, \tag{3.79}$$

and consequently, we look for the $u_n(r)$ satisfying

$$u_n(r) = \mathrm{O}\big(1/\sqrt{r}\big) \quad \text{and} \quad d_r u_n(r) - \mathrm{i}ku_n(r) = \mathrm{o}\big(1/\sqrt{r}\big) \quad \text{as } r \to \infty. \tag{3.80}$$

We know (see, e.g., p. 789 in [26]) that

$$\begin{aligned} J_n(r) &\sim \frac{\cos(r - \pi/4 - n\pi/2)}{\sqrt{\pi r/2}}, \quad & Y_n(r) &\sim \frac{\sin(r - \pi/4 - n\pi/2)}{\sqrt{\pi r/2}}, \quad \text{as } r \to \infty, \\ d_r J_n(r) &\sim -\frac{\sin(r - \pi/4 - n\pi/2)}{\sqrt{\pi r/2}}, \quad & d_r Y_n(r) &\sim \frac{\cos(r - \pi/4 - n\pi/2)}{\sqrt{\pi r/2}}, \quad \text{as } r \to \infty, \end{aligned} \tag{3.81}$$

and therefore, $v_{n,1}(r)$, $v_{n,2}(r)$ in (3.30), and also $v_n(r)$ in (3.34) and (3.35), satisfy only the first condition in (3.80). On the other hand, from (3.37) and (3.81) we get

$$\begin{aligned} H_n^{(1)}(r) &\sim \frac{\mathrm{e}^{\mathrm{i}(r-\pi/4-n\pi/2)}}{\sqrt{\pi r/2}}, \quad & H_n^{(2)}(r) &\sim \frac{\mathrm{e}^{-\mathrm{i}(r-\pi/4-n\pi/2)}}{\sqrt{\pi r/2}}, \quad \text{as } r \to \infty, \\ d_r H_n^{(1)}(r) &\sim \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}(r-\pi/4-n\pi/2)}}{\sqrt{\pi r/2}}, \quad & d_r H_n^{(2)}(r) &\sim \frac{-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}(r-\pi/4-n\pi/2)}}{\sqrt{\pi r/2}}, \quad \text{as } r \to \infty, \end{aligned} \tag{3.82}$$

and then

$$
\begin{aligned}
d_r H_n^{(1)}(r) - \mathrm{i} H_n^{(1)}(r) &= \mathrm{o}\left(\frac{\mathrm{e}^{\mathrm{i}(r-\pi/4-n\pi/2)}}{\sqrt{\pi r/2}}\right), \quad \text{as } r \to \infty, \\
d_r H_n^{(2)}(r) - \mathrm{i} H_n^{(2)}(r) &\sim -2\mathrm{i}\,\frac{\mathrm{e}^{-\mathrm{i}(r-\pi/4-n\pi/2)}}{\sqrt{\pi r/2}}, \quad \text{as } r \to \infty.
\end{aligned} \tag{3.83}
$$

Therefore, both

$$
w_n(r) = H_n^{(1)}(kr)
$$

and $v_n(r)$ given by (3.40), namely

$$
v_n(r) = -\frac{\mathrm{i}\pi}{4}\int_R^r \rho H_n^{(1)}(kr) H_n^{(2)}(k\rho) f_n(\rho)\,\mathrm{d}\rho - \frac{\mathrm{i}\pi}{4}\int_r^\infty \rho H_n^{(2)}(kr) H_n^{(1)}(k\rho) f_n(\rho)\,\mathrm{d}\rho, \tag{3.84}
$$

satisfy the conditions at infinity (3.80). From (3.37) and (3.62) we get

$$
d_r H_n^{(1)}(r) = \frac{1}{2}\left[H_{n-1}^{(1)}(r) - H_{n+1}^{(1)}(r)\right] \quad \text{and} \quad d_r H_n^{(2)}(r) = \frac{1}{2}\left[H_{n-1}^{(2)}(r) - H_{n+1}^{(2)}(r)\right], \tag{3.85}
$$

and then,

$$
d_r H_n^{(1)}(kr) = \frac{k}{2}\left[H_{n-1}^{(1)}(kr) - H_{n+1}^{(1)}(kr)\right] \quad \text{and} \quad d_r H_n^{(2)}(r) = \frac{k}{2}\left[H_{n-1}^{(2)}(kr) - H_{n+1}^{(2)}(kr)\right], \tag{3.86}
$$

which are used in the calculation of the derivatives in proposition 3.3. In this way we get:

Corollary 3.5. The Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ of the Helmholtz equation $Lu \equiv \Delta u + k^2 u = f$ in the exterior of a closed disc of radius R with its center at the origin are given by

$$
u_n(r) = v_n(r) + \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)}\left[g_n + \frac{\mathrm{i}\pi}{4}\int_R^\infty \rho H_n^{(2)}(kR) H_n^{(1)}(k\rho) f_n(\rho)\,\mathrm{d}\rho\right], \tag{3.87}
$$

if $H_n^{(1)}(kR) \neq 0$ for any n, for the Dirichlet problem (2.1), (2.2), and as

$$
\begin{aligned}
u_n(r) = v_n(r) &- \frac{H_n^{(1)}(kr)}{H_{n-1}^{(1)}(kR) - H_{n+1}^{(1)}(kR)} \\
&\times \left\{\frac{2}{k}h_n - \frac{\mathrm{i}\pi}{4}\int_R^\infty \rho\left[H_{n-1}^{(2)}(kR) - H_{n+1}^{(2)}(kR)\right] H_n^{(1)}(k\rho) f_n(\rho)\,\mathrm{d}\rho\right\},
\end{aligned} \tag{3.88}
$$

if $H_{n-1}^{(1)}(kR) \neq H_{n+1}^{(1)}(kR)$ for any n, for the Neumann problem (2.1), (2.3), where $v_n(r)$ is given by (3.84).

3.2.3. Helmholtz equation (monotone case)

The operator for this equation is $Lu \equiv \Delta u - k^2 u$. For this operator associated with the Dirichlet problem (2.1), (2.2) and the Neumann problem (2.1), (2.3),

$$u\big(r\mathrm{e}^{\mathrm{i}\theta}\big) = \mathrm{o}(1) \quad \text{as } r \to \infty,$$

and therefore, we look for the $u_n(r)$ satisfying

$$u_n(r) = \mathrm{o}(1) \quad \text{as } r \to \infty. \tag{3.89}$$

From (3.46) and (3.81) we obtain the following asymptotic behavior as $r \to \infty$.

$$I_n(r) \sim \frac{\mathrm{e}^r}{\sqrt{2\pi r}}, \qquad K_n(r) \sim \frac{\mathrm{e}^{-r}}{\sqrt{2\pi r}}, \qquad d_r I_n(r) \sim \frac{\mathrm{e}^r}{\sqrt{2\pi r}}, \qquad d_r K_n(r) \sim \frac{\mathrm{e}^{-r}}{\sqrt{2\pi r}}, \tag{3.90}$$

and therefore, we take

$$w_n(r) = K_n(kr)$$

and $v_n(r)$ from (3.49)

$$v_n(r) = -\int_R^r \rho K_n(kr) I_n(k\rho) f_n(\rho)\,\mathrm{d}\rho - \int_r^\infty \rho I_n(kr) K_n(k\rho) f_n(\rho)\,\mathrm{d}\rho, \tag{3.91}$$

which satisfy the conditions at infinity. Using again (3.68) to calculate the derivatives used in proposition 3.3 for the Neumann conditions, we have:

Corollary 3.6. The Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ of the Helmholtz equation $Lu \equiv \Delta u - k^2 u = f$ in the exterior of a closed disc of radius R with its center at the origin are given by

$$u_n(r) = v_n(r) + \frac{K_n(kr)}{K_n(kR)}\left[g_n + \int_R^\infty \rho I_n(kR) K_n(k\rho) f_n(\rho)\,\mathrm{d}\rho\right], \tag{3.92}$$

if $K_n(kR) \neq 0$ for any n, for the Dirichlet problem (2.1), (2.2), and as

$$u_n(r) = v_n(r) + \frac{K_n(kr)}{K_{n-1}(kR) + K_{n+1}(kR)} \times \left\{\frac{2}{k}h_n - \int_R^\infty \rho\big[I_{n-1}(kR) + I_{n+1}(kR)\big] K_n(k\rho) f_n(\rho)\,\mathrm{d}\rho\right\}, \tag{3.93}$$

if $K_{n-1}(kR) + K_{n+1}(kR) \neq 0$ for any n, for the Neumann problem (2.1), (2.3), where $v_n(r)$ is given by (3.91).

3.3. Annular domains

For the annulus with the radii $0 < R_1 < R_2 < \infty$, we look for a solution of equation (3.1) of the form

$$u_n(r) = v_n(r) + c_n(R_1, R_2) v_{n,1}(r) + d_n(R_1, R_2) v_{n,2}(r), \quad R_1 < r < R_2, \tag{3.94}$$

where $v_n(r)$ is one of the two particular solutions of equation (3.1) given by (3.21) or (3.22), and $v_{n,1}(r)$ and $v_{n,2}(r)$ are the two linearly independent solutions of the homogeneous form of the same equation. Since, in this case, we do not have conditions at zero or at infinity, $v_n(r)$ can be any of the solutions in (3.21) or (3.22). Consequently, to have (2.14), we can calculate $c_n(R_1, R_2)$ and $d_n(R_1, R_2)$ from the system

$$\begin{aligned} c_n(R_1, R_2)v_{n,1}(R_1) + d_n(R_1, R_2)v_{n,2}(R_1) &= g_n^{(1)} - v_n(R_1), \\ c_n(R_1, R_2)v_{n,1}(R_2) + d_n(R_1, R_2)v_{n,2}(R_2) &= g_n^{(2)} - v_n(R_2) \end{aligned} \tag{3.95}$$

for a Dirichlet problem, and to have (2.15), we get $c_n(R_1, R_2)$ and $d_n(R_1, R_2)$ from

$$\begin{aligned} c_n(R_1, R_2)d_r v_{n,1}(R_1) + d_n(R_1, R_2)d_r v_{n,2}(R_1) &= -h_n^{(1)} - d_r v_n(R_1), \\ c_n(R_1, R_2)d_r v_{n,1}(R_2) + d_n(R_1, R_2)d_r v_{n,2}(R_2) &= h_n^{(2)} - d_r v_n(R_2) \end{aligned} \tag{3.96}$$

for a Neumann problem. Also, we can look for a solution which satisfies Dirichlet conditions on the circle of radius R_1 and Neumann conditions on the circle of radius R_2 (or inversely),

$$\begin{aligned} c_n(R_1, R_2)v_{n,1}(R_1) + d_n(R_1, R_2)v_{n,2}(R_1) &= g_n - v_n(R_1), \\ c_n(R_1, R_2)d_r v_{n,1}(R_2) + d_n(R_1, R_2)d_r v_{n,2}(R_2) &= h_n - d_r v_n(R_2). \end{aligned} \tag{3.97}$$

As in the previous cases of domains we can state:

Proposition 3.4. Under the conditions of theorem 2.1, the Fourier coefficients $u_n(r)$ of the solution $u(r, \theta)$ of (2.1) in an annulus centered at the origin and bounded by the radii R_1 and R_2 $(0 < R_1 < R_2 < \infty)$ are given by

$$\begin{aligned} u_n(r) = v_n(r) &+ \frac{v_{n,1}(R_2)v_{n,2}(r) - v_{n,2}(R_2)v_{n,1}(r)}{D_n^{(1)}(R_1, R_2)}\left[v_n(R_1) - g_n^{(1)}\right] \\ &+ \frac{v_{n,2}(R_1)v_{n,1}(r) - v_{n,1}(R_1)v_{n,2}(r)}{D_n^{(1)}(R_1, R_2)}\left[v_n(R_2) - g_n^{(2)}\right], \quad R_1 < r < R_2, \end{aligned} \tag{3.98}$$

if $D_n^{(1)}(R_1, R_2) = v_{n,1}(R_1)v_{n,2}(R_2) - v_{n,1}(R_2)v_{n,2}(R_1) \neq 0$, for the Dirichlet problem (2.1), (2.2), and by

$$\begin{aligned} u_n(r) = v_n(r) &+ \frac{d_r v_{n,1}(R_2)v_{n,2}(r) - d_r v_{n,2}(R_2)v_{n,1}(r)}{D_n^{(2)}(R_1, R_2)}\left[d_r v_n(R_1) + h_n^{(1)}\right] \\ &+ \frac{d_r v_{n,2}(R_1)v_{n,1}(r) - d_r v_{n,1}(R_1)v_{n,2}(r)}{D_n^{(2)}(R_1, R_2)}\left[d_r v_n(R_2) - h_n^{(2)}\right], \quad R_1 < r < R_2, \end{aligned} \tag{3.99}$$

if $D_n^{(2)}(R_1, R_2) = d_r v_{n,1}(R_1)d_r v_{n,2}(R_2) - d_r v_{n,1}(R_2)d_r v_{n,2}(R_1) \neq 0$, for the Neumann problem (2.1), (2.3). Also, for the problem associated with equation (2.1) and Dirichlet condition (2.2) on the circle of radius R_1 and Neumann condition (2.3) on that of radius R_2, we have

$$u_n(r) = v_n(r) + \frac{d_r v_{n,1}(R_2)v_{n,2}(r) - d_r v_{n,2}(R_2)v_{n,1}(r)}{D_n^{(0)}(R_1, R_2)}\big[v_n(R_1) - g_n\big] + \frac{v_{n,2}(R_1)v_{n,1}(r) - v_{n,1}(R_1)v_{n,2}(r)}{D_n^{(0)}(R_1, R_2)}\big[d_r v_n(R_2) - h_n\big], \quad R_1 < r < R_2, \tag{3.100}$$

if $D_n^{(0)}(R_1, R_2) = v_{n,1}(R_1)d_r v_{n,2}(R_2) - d_r v_{n,1}(R_2)v_{n,2}(R_1) \neq 0$. Here, $v_{n,1}(r)$ and $v_{n,1}(r)$ are two linearly independent solutions of the homogeneous form of equation (3.1), and $v_n(r)$ is given by either (3.21) or (3.22).

Remark 3.6. Interchanging R_1 and R_2 and replacing h_n by $-h_n$ in (3.100), we find that the Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ for the problem associated with equation (2.1) and conditions (2.2) on the circle of radius R_2 and (2.3) on that of radius R_1 are given by

$$u_n(r) = v_n(r) + \frac{d_r v_{n,1}(R_1)v_{n,2}(r) - d_r v_{n,2}(R_1)v_{n,1}(r)}{D_n^{(0)}(R_2, R_1)}\big[v_n(R_2) - g_n\big] + \frac{v_{n,2}(R_2)v_{n,1}(r) - v_{n,1}(R_2)v_{n,2}(r)}{D_n^{(0)}(R_2, R_1)}\big[d_r v_n(R_1) + h_n\big], \quad R_2 < r < R_1, \tag{3.101}$$

if $D_n^{(0)}(R_2, R_1) \neq 0$.

Below we apply the above proposition to construct solutions of the three problems in an annulus.

3.3.1. Poisson equation

Either (3.25)–(3.26) or (3.27)–(3.28) can be used for $v_n(r)$ in proposition 3.4. The results obtained for u_n by applying proposition 3.4 will be the same in both cases. Using $v_{n,1}(r)$ and $v_{n,2}(r)$ from (3.24), we first get

$$D_0^{(1)}(R_1, R_2) = \log(R_2/R_1), \qquad D_n^{(1)}(R_1, R_2) = (R_1/R_2)^{|n|} - (R_2/R_1)^{|n|} \quad \text{for } n \neq 0, \tag{3.102}$$

$$D_0^{(2)}(R_1, R_2) = 0, \qquad D_n^{(2)}(R_1, R_2) = \frac{n^2}{R_1 R_2}\left[\left(\frac{R_2}{R_1}\right)^{|n|} - \left(\frac{R_1}{R_2}\right)^{|n|}\right] \quad \text{for } n \neq 0, \tag{3.103}$$

and

$$D_0^{(0)}(R_1, R_2) = \frac{1}{R_2}, \qquad D_n^{(0)}(R_1, R_2) = \frac{-|n|}{R_2}\left[\left(\frac{R_1}{R_2}\right)^{|n|} + \left(R_2/R_1\right)^{|n|}\right] \quad \text{for } n \neq 0. \tag{3.104}$$

As before, the solution of the Neumann problem associated with the Laplace operator is unique up to an additive constant for $n = 0$. Since $d_r v_{0,1}(r) = 0$ and $h_0^{(2)} R_2 + h_0^{(1)} R_1 = \int_{R_1}^{R_2} \rho f_0(\rho)\, \mathrm{d}\rho$, which follows from (2.4), we obtain from (3.96) that $c_0(R_1, R_2)$ is an arbitrary constant and the constant $d_0(R_1, R_2)$ is uniquely determined. Therefore, from proposition (3.4) we get the following corollary.

Corollary 3.7. The Fourier coefficients $u_n(r)$ of the solution $u(r,\theta)$ of the Poisson equation in an annulus centered at the origin and bounded by the radii R_1 and R_2 $(0 < R_1 < R_2 < \infty)$ are given by

$$\begin{aligned} u_0(r) &= v_0(r) - \frac{\log(R_2/r)}{\log(R_2/R_1)}\big[v_0(R_1) - g_0^{(1)}\big] - \frac{\log(R_1/r)}{\log(R_1/R_2)}\big[v_0(R_2) - g_0^{(2)}\big], \\ u_n(r) &= v_n(r) - \frac{(R_2/r)^{|n|} - (r/R_2)^{|n|}}{(R_2/R_1)^{|n|} - (R_1/R_2)^{|n|}}\big[v_n(R_1) - g_n^{(1)}\big] \\ &\quad - \frac{(R_1/r)^{|n|} - (r/R_1)^{|n|}}{(R_1/R_2)^{|n|} - (R_2/R_1)^{|n|}}\big[v_n(R_2) - g_n^{(2)}\big], \quad \text{for } n \neq 0, \end{aligned} \tag{3.105}$$

for the Dirichlet problem (2.1), (2.2), and as

$$\begin{aligned} u_0(r) &= v_0(r) - R_1 \log(r)\big[h_0^{(1)} + d_r v_0(R_1)\big] + c_0 \\ &= v_0(r) + R_2 \log(r)\big[h_0^{(2)} - d_r v_0(R_2)\big] + c_0, \\ u_n(r) &= v_n(r) + \frac{(R_2/r)^{|n|} + (r/R_2)^{|n|}}{(R_2/R_1)^{|n|} - (R_1/R_2)^{|n|}} \frac{R_1}{|n|}\big[d_r v_n(R_1) + h_n^{(1)}\big] \\ &\quad + \frac{(R_1/r)^{|n|} + (r/R_1)^{|n|}}{(R_1/R_2)^{|n|} - (R_2/R_1)^{|n|}} \frac{R_2}{|n|}\big[d_r v_n(R_2) - h_n^{(2)}\big], \quad \text{for } n \neq 0, \end{aligned} \tag{3.106}$$

for the Neumann problem (2.1), (2.3). In the above equation, c_0 is an arbitrary real constant. For the solution of the Poisson equation subject to Dirichlet data (2.2) on $r = R_1$ and Neumann data (2.3) on $r = R_2$, we have

$$\begin{aligned} u_0(r) &= v_0(r) - \big[v_0(R_1) - g_0\big] + R_2 \log(R_1/r)\big[d_r v_0(R_2) - h_0\big], \\ u_n(r) &= v_n(r) - \frac{(R_2/r)^{|n|} + (r/R_2)^{|n|}}{(R_2/R_1)^{|n|} + (R_1/R_2)^{|n|}}\big[v_n(R_1) - g_n\big] \\ &\quad + \frac{(R_1/r)^{|n|} - (r/R_1)^{|n|}}{(R_1/R_2)^{|n|} + (R_2/R_1)^{|n|}} \frac{R_2}{|n|}\big[d_r v_n(R_2) - h_n\big], \quad \text{for } n \neq 0. \end{aligned} \tag{3.107}$$

In (3.105)–(3.107), $v_0(r)$ and $v_n(r)$ can be taken from either (3.25)–(3.26) or (3.27)–(3.28). If we use (3.25)–(3.26), we have

$$\begin{aligned} d_r v_0(R_1) &= \frac{1}{R_1 \log(R_1)} v_0(R_1), \qquad & d_r v_0(R_2) &= 0, \\ d_r v_n(R_1) &= \frac{-|n|}{R_1} v_n(R_1), & d_r v_n(R_2) &= \frac{|n|}{R_2} v_n(R_2), \quad \text{for } n \neq 0, \end{aligned} \tag{3.108}$$

and if we use (3.27)–(3.28), we have

$$
\begin{aligned}
& d_r v_0(R_1) = 0, && d_r v_0(R_2) = \frac{1}{R_2 \log(R_2)} v_0(R_2), \\
& d_r v_n(R_1) = \frac{|n|}{R_1} v_n(R_1), && d_r v_n(R_2) = \frac{-|n|}{R_2} v_n(R_2), \quad \text{for } n \neq 0.
\end{aligned}
\tag{3.109}
$$

Remark 3.7. For the solution of the Poisson equation subject to Dirichlet data (2.2) on $r = R_2$ and Neumann data (2.3) on $r = R_1$, instead of (3.107), we have

$$
\begin{aligned}
u_0(r) &= v_0(r) - \left[v_0(R_2) - g_0\right] + R_1 \log(R_2/r)\left[d_r v_0(R_1) + h_0\right], \\
u_n(r) &= v_n(r) - \frac{(R_1/r)^{|n|} + (r/R_1)^{|n|}}{(R_1/R_2)^{|n|} + (R_2/R_1)^{|n|}}\left[v_n(R_2) - g_n\right] \\
&\quad + \frac{(R_2/r)^{|n|} - (r/R_2)^{|n|}}{(R_2/R_1)^{|n|} + (R_1/R_2)^{|n|}} \frac{R_1}{|n|}\left[d_r v_n(R_1) + h_n\right], \quad \text{for } n \neq 0.
\end{aligned}
\tag{3.110}
$$

3.3.2. Helmholtz equation (oscillatory case)

The operator for this equation is $Lu \equiv \Delta u + k^2 u$. Here either (3.34) or (3.35) can be used for $v_n(r)$ in the proposition 3.4. Using $v_{n,1}(r)$ and $v_{n,2}(r)$ given by (3.30) we first get

$$
D_n^{(1)}(R_1, R_2) = J_n(kR_1)Y_n(kR_2) - J_n(kR_2)Y_n(kR_1), \tag{3.111}
$$

$$
D_n^{(2)}(R_1, R_2) = d_r J_n(kR_1) d_r Y_n(kR_2) - d_r J_n(kR_2) d_r Y_n(kR_1), \tag{3.112}
$$

and

$$
D_n^{(0)}(R_1, R_2) = J_n(kR_1) d_r Y_n(kR_2) - d_r J_n(kR_2) Y_n(kR_1). \tag{3.113}
$$

Now, we get the following corollary from proposition 3.4.

Corollary 3.8. The Fourier coefficients $u_n(r)$ of the solution $u(r, \theta)$ of the Helmholtz equation $Lu \equiv \Delta u + k^2 u = f$ in an annulus centered at the origin and bounded by the radii R_1 and R_2 ($0 < R_1 < R_2 < \infty$) are given by

$$
\begin{aligned}
u_n(r) &= v_n(r) - \frac{J_n(kR_2)Y_n(kr) - Y_n(kR_2)J_n(kr)}{J_n(kR_2)Y_n(kR_1) - Y_n(kR_2)J_n(kR_1)}\left[v_n(R_1) - g_n^{(1)}\right] \\
&\quad - \frac{Y_n(kR_1)J_n(kr) - J_n(kR_1)Y_n(kr)}{Y_n(kR_1)J_n(kR_2) - J_n(kR_1)Y_n(kR_2)}\left[v_n(R_2) - g_n^{(2)}\right],
\end{aligned}
\tag{3.114}
$$

if $D_n^{(1)}(R_1, R_2)$ in (3.111) is not zero for any n, for the Dirichlet problem (2.1), (2.2), and as

$$u_n(r) = v_n(r) - \frac{d_r J_n(kR_2) Y_n(kr) - d_r Y_n(kR_2) J_n(kr)}{d_r J_n(kR_2) d_r Y_n(kR_1) - d_r Y_n(kR_2) d_r J_n(kR_1)} \left[d_r v_n(R_1) + h_n^{(1)} \right] - \frac{d_r Y_n(kR_1) J_n(kr) - d_r J_n(kR_1) Y_n(kr)}{d_r Y_n(kR_1) d_r J_n(kR_2) - d_r J_n(kR_1) d_r Y_n(kR_2)} \left[d_r v_n(R_2) - h_n^{(2)} \right], \quad R_1 < r < R_2, \tag{3.115}$$

if $D_n^{(2)}(R_1, R_2)$ in (3.112) is not zero for any n, for the Neumann problem (2.1), (2.3). For the solution of Helmholtz equation $Lu \equiv \Delta u + k^2 u = f$ subject to Dirichlet data (2.2) on $r = R_1$ and Neumann data (2.3) on $r = R_2$, we have

$$u_n(r) = v_n(r) - \frac{d_r J_n(kR_2) Y_n(kr) - d_r Y_n(kR_2) J_n(kr)}{d_r J_n(kR_2) Y_n(kR_1) - d_r Y_n(kR_2) J_n(kR_1)} \left[v_n(R_1) - g_n \right] - \frac{Y_n(kR_1) J_n(kr) - J_n(kR_1) Y_n(kr)}{Y_n(kR_1) d_r J_n(kR_2) - J_n(kR_1) d_r Y_n(kR_2)} \left[d_r v_n(R_2) - h_n \right], \quad R_1 < r < R_2, \tag{3.116}$$

if $D_n^{(0)}(R_1, R_2)$ in (3.113) is not zero for any n. In (3.114)–(3.116), $v_n(r)$ can be taken from either (3.34) or (3.35). Also, we have

$$d_r v_n(R_1) = \frac{d_r Y_n(kR_1)}{Y_n(kR_1)} v_n(R_1), \qquad d_r v_n(R_2) = \frac{d_r J_n(kR_2)}{J_n(kR_2)} v_n(R_2) \tag{3.117}$$

if we use (3.34), and

$$d_r v_n(R_1) = \frac{d_r J_n(kR_1)}{J_n(kR_1)} v_n(R_1), \qquad d_r v_n(R_2) = \frac{d_r Y_n(kR_2)}{Y_n(kR_2)} v_n(R_2) \tag{3.118}$$

if we use (3.35). The derivatives of $J_n(kr)$ and $Y_n(kr)$ can be calculated using (3.63).

Remark 3.8. For the solution of Helmholtz equation $Lu \equiv \Delta u + k^2 u = f$ subject to Dirichlet data (2.2) on $r = R_2$ and Neumann data (2.3) on $r = R_1$, instead of (3.116), we have

$$u_n(r) = v_n(r) - \frac{d_r J_n(kR_1) Y_n(kr) - d_r Y_n(kR_1) J_n(kr)}{d_r J_n(kR_1) Y_n(kR_2) - d_r Y_n(kR_1) J_n(kR_2)} \left[v_n(R_2) - g_n \right] - \frac{Y_n(kR_2) J_n(kr) - J_n(kR_2) Y_n(kr)}{Y_n(kR_2) d_r J_n(kR_1) - J_n(kR_2) d_r Y_n(kR_1)} \left[d_r v_n(R_1) + h_n \right], \quad R_1 < r < R_2. \tag{3.119}$$

3.3.3. Helmholtz equation (monotone case)

The operator for this equation is $Lu \equiv \Delta u - k^2 u$. Similar to the other Helmholtz equation, using $v_{n,1}(r)$ and $v_{n,2}(r)$ from (3.45) we obtain

$$D_n^{(1)}(R_1, R_2) = I_n(kR_1) K_n(kR_2) - I_n(kR_2) K_n(kR_1), \tag{3.120}$$

$$D_n^{(2)}(R_1, R_2) = d_r I_n(kR_1) d_r K_n(kR_2) - d_r I_n(kR_2) d_r K_n(kR_1), \tag{3.121}$$

and

$$D_n^{(0)}(R_1, R_2) = I_n(kR_1) d_r K_n(kR_2) - d_r I_n(kR_2) K_n(kR_1). \tag{3.122}$$

Now, from proposition 3.4 we obtain:

Corollary 3.9. The Fourier coefficients $u_n(r)$ of the solution $u(r, \theta)$ of the Helmholtz equation $Lu \equiv \Delta u - k^2 u = f$ in an annulus centered at the origin and bounded by the radii R_1 and R_2 ($0 < R_1 < R_2 < \infty$) are given by

$$\begin{aligned} u_n(r) = v_n(r) &- \frac{I_n(kR_2)K_n(kr) - K_n(kR_2)I_n(kr)}{I_n(kR_2)K_n(kR_1) - K_n(kR_2)I_n(kR_1)}\big[v_n(R_1) - g_n^{(1)}\big] \\ &- \frac{K_n(kR_1)I_n(kr) - I_n(kR_1)K_n(kr)}{K_n(kR_1)I_n(kR_2) - I_n(kR_1)K_n(kR_2)}\big[v_n(R_2) - g_n^{(2)}\big], \quad R_1 < r < R_2, \end{aligned} \tag{3.123}$$

if $D_n^{(1)}(R_1, R_2)$ in (3.120) is not zero for any n, for the Dirichlet problem (2.1), (2.2), and as

$$\begin{aligned} u_n(r) = v_n(r) &- \frac{d_r I_n(kR_2)K_n(kr) - d_r K_n(kR_2)I_n(kr)}{d_r I_n(kR_2)d_r K_n(kR_1) - d_r K_n(kR_2)d_r I_n(kR_1)}\big[d_r v_n(R_1) + h_n^{(1)}\big] \\ &- \frac{d_r K_n(kR_1)I_n(kr) - d_r I_n(kR_1)K_n(kr)}{d_r K_n(kR_1)d_r I_n(kR_2) - d_r I_n(kR_1)d_r K_n(kR_2)}\big[d_r v_n(R_2) - h_n^{(2)}\big], \\ R_1 < r < R_2, \end{aligned} \tag{3.124}$$

if $D_n^{(2)}(R_1, R_2)$ in (3.121) is not zero for any n, for the Neumann problem (2.1), (2.3). For the solution of Helmholtz equation $Lu \equiv \Delta u - k^2 u = f$ subject to Dirichlet data (2.2) on $r = R_1$ and Neumann data (2.3) on $r = R_2$, we have

$$\begin{aligned} u_n(r) = v_n(r) &- \frac{d_r I_n(kR_2)K_n(kr) - d_r K_n(kR_2)I_n(kr)}{d_r I_n(kR_2)K_n(kR_1) - d_r K_n(kR_2)I_n(kR_1)}\big[v_n(R_1) - g_n\big] \\ &- \frac{K_n(kR_1)I_n(kr) - I_n(kR_1)K_n(kr)}{K_n(kR_1)d_r I_n(kR_2) - I_n(kR_1)d_r K_n(kR_2)}\big[d_r v_n(R_2) - h_n\big], \quad R_1 < r < R_2, \end{aligned} \tag{3.125}$$

if $D_n^{(0)}(R_1, R_2)$ in (3.122) is not zero for any n. In (3.123)–(3.125), $v_n(r)$ can be taken from either (3.48) or (3.49). Also, we have

$$d_r v_n(R_1) = \frac{d_r K_n(kR_1)}{K_n(kR_1)} v_n(R_1), \qquad d_r v_n(R_2) = \frac{d_r I_n(kR_2)}{I_n(kR_2)} v_n(R_2) \tag{3.126}$$

if we use (3.48), and

$$d_r v_n(R_1) = \frac{d_r I_n(kR_1)}{I_n(kR_1)} v_n(R_1), \qquad d_r v_n(R_2) = \frac{d_r K_n(kR_2)}{K_n(kR_2)} v_n(R_2) \tag{3.127}$$

if we use (3.49). The derivatives of $I_n(kr)$ and $K_n(kr)$ can be calculated using (3.68).

Remark 3.9. For the solution of Helmholtz equation $Lu \equiv \Delta u - k^2 u = f$ subject to Dirichlet data (2.2) on $r = R_2$ and Neumann data (2.3) on $r = R_1$, instead of (3.125), we have

$$u_n(r) = v_n(r) - \frac{d_r I_n(kR_1)K_n(kr) - d_r K_n(kR_1)I_n(kr)}{d_r I_n(kR_1)K_n(kR_2) - d_r K_n(kR_1)I_n(kR_2)}\big[v_n(R_2) - g_n\big] - \frac{K_n(kR_2)I_n(kr) - I_n(kR_2)K_n(kr)}{K_n(kR_2)d_r I_n(kR_1) - I_n(kR_2)d_r K_n(kR_1)}\big[d_r v_n(R_1) + h_n\big], \quad R_1 < r < R_2. \tag{3.128}$$

4. Description of the numerical algorithm for the solution of the two-dimensional problems

The numerical algorithm for the two-dimensional problems presented below is based on the FFT and an integral representation of exact solutions of various one-dimensional problems discussed in the previous section. In all the one-dimensional problems considered in the previous section, we have $\alpha_n(r) = 1$ and $D_n(r)$ is a constant multiple of $1/r$. (It may be worth recalling that $\alpha_n(r)$ is the coefficient multiplying the second derivative term (see (3.4)) and $D_n(r)$ is the Wronskian of two linearly independent solutions (see (3.16))). Therefore, we consider below only this case. However, it should be pointed out here that all that follows below can also be applied to other problems with more complicated expressions of $\alpha_n(r)$ and $D_n(r)$ without any loss of efficiency or accuracy.

The fast stable algorithm 4.2 given below for the two-dimensional problems requires a fast stable algorithm (either algorithm 4.1A or algorithm 4.1B depending on the case, see below) for the evaluation of one-dimensional integrals, which we discuss first, followed by the treatment of the algorithm for the two-dimensional problems.

4.1. Algorithm for the one-dimensional integrals

Computation of the solutions of various one-dimensional problems given in the corollaries of the previous section requires evaluation of integrals of the form (3.21) or (3.22) at discretization points. With our choices of linearly independent solutions $v_{n,1}$ and $v_{n,2}$ given in section 3 for all three operators, it is worth recalling the following: (a) Choice of (3.22) is the appropriate one for interior and exterior problems for all three operators except for the case of the exterior problem associated with the oscillatory Helmholtz operator when (3.21), and not (3.22), is the appropriate choice for v_n; (b) For the problems in an annular region, either of the two choices, (3.21) and (3.22), is appropriate to calculate v_n. This is based on purely theoretical consideration. However, numerical considerations limit these choices even further as we will see below. In particular, for numerical stability of the fast algorithm (presented below) for computing v_n, (3.22) is preferable over (3.21) as we will see.

Therefore, except in one case (exterior oscillatory Helmholtz), the form (3.22) is the one to be used in computations of solutions, which require evaluating expressions of the form

$$Q_n(r) = Q_n^{(1)}(r) + Q_n^{(2)}(r) = \int_{R_1}^{r} \rho v_{n,2}(r) v_{n,1}(\rho) f_n(\rho)\, \mathrm{d}\rho + \int_{r}^{R_2} \rho v_{n,1}(r) v_{n,2}(\rho) f_n(\rho)\, \mathrm{d}\rho. \tag{4.1}$$

In the case of the exterior problem associated with the oscillatory Helmholtz operator, representation (3.21) for v_n is the one to be used and requires evaluating expressions of the form

$$S_n(r) = S_n^{(1)}(r) + S_n^{(2)}(r) = \int_{R_1}^{r} \rho v_{n,1}(r) v_{n,2}(\rho) f_n(\rho)\, \mathrm{d}\rho + \int_{r}^{R_2} \rho v_{n,2}(r) v_{n,1}(\rho) f_n(\rho)\, \mathrm{d}\rho. \tag{4.2}$$

When the domain Ω is the exterior of a closed disc, a large but finite value for R_2 is used in the integrals above. As we have seen in the previous section, two linearly independent solutions $v_{n,1}(r)$ and $v_{n,2}(r)$ of the homogeneous form of the equation (3.1) have opposite limit behaviors (i.e. when one goes to zero, the other one goes to infinity) at each of the extremes: $r \to 0$, and $r \to \infty$. Consequently, for numerical stability in computing (4.1) and (4.2), we have included $v_{n,1}(r)$ and $v_{n,2}(r)$ under the integrals.

In [13], a fast stable algorithm has been proposed for computing the integrals of the type $Q_n(r)$ in (4.1) arising from a problem associated with Cauchy–Riemann equations. We extend that idea to a more general case here. For a discretization $R_1 = r_1 < r_2 < \cdots < r_M = R_2$, not necessarily equidistant, of the interval $[R_1, R_2]$, it is worthwhile to give the following algorithms for stable computations of the integrals $S_n(r)$ and $Q_n(r)$ at the points of this discretization for values of $n \geqslant 0$. These computed values can, in principle, then be used to evaluate these integrals for $n < 0$ at the discretization points using relations (4.6) for Q_n and (4.12) for S_n, which are given below after algorithms 4.1A and 4.1B, respectively.

Algorithm 4.1A – Sequential algorithm for the integrals Q_n.
Step 1. Compute $Q_n^{(1)}(r_m)$, $m = 2, \ldots, M$, as

$$Q_n^{(1)}(r_2) = \int_{r_1}^{r_2} \rho v_{n,2}(r_2) v_{n,1}(\rho) f_n(\rho)\, \mathrm{d}\rho,$$
$$Q_n^{(1)}(r_m) = \frac{v_{n,2}(r_m)}{v_{n,2}(r_{m-1})} Q_n^{(1)}(r_{m-1}) + \int_{r_{m-1}}^{r_m} \rho v_{n,2}(r_m) v_{n,1}(\rho) f_n(\rho)\, \mathrm{d}\rho, \quad m = 3, \ldots, M. \tag{4.3}$$

Step 2. Compute $Q_n^{(2)}(r_m),\ m = M-1, \dots, 1$, as

$$\begin{aligned} Q_n^{(2)}(r_{M-1}) &= \int_{r_{M-1}}^{r_M} \rho v_{n,1}(r_{M-1}) v_{n,2}(\rho) f_n(\rho)\,\mathrm{d}\rho, \\ Q_n^{(2)}(r_m) &= \frac{v_{n,1}(r_m)}{v_{n,1}(r_{m+1})} Q_n^{(2)}(r_{m+1}) + \int_{r_m}^{r_{m+1}} \rho v_{n,1}(r_m) v_{n,2}(\rho) f_n(\rho)\,\mathrm{d}\rho, \\ & m = M-2, \dots, 1. \end{aligned} \tag{4.4}$$

Step 3. Compute $Q_n(r_m),\ m = 1, \dots, M$, as

$$\begin{aligned} &Q_n(r_1) = Q_n^{(2)}(r_1), \qquad Q_n(r_M) = Q_n^{(1)}(r_M), \\ &Q_n(r_m) = Q_n^{(1)}(r_m) + Q_n^{(2)}(r_m), \quad m = 2, \dots, M-1. \end{aligned} \tag{4.5}$$

It is easily verified that the coefficients in front of $Q_n^{(1)}(r_{m-1})$ and $Q_n^{(2)}(r_{m-1})$ in the second equation of (4.3) and that of (4.4) are less than one for the Poisson equation and hence the computational process in the above algorithm is stable. For this stability reason, the above algorithm should be used for solving the Poisson equation in all three different types of domains.

For the oscillatory Helmholtz equation, due to the oscillatory nature of the linearly independent solutions, this coefficient is an oscillatory function of r_m and thus takes values above or below one depending on the value of r_m in the above recursion formulae. Thus, in this case also, the computational process in the above algorithm is stable and, hence, the above algorithm should be used for both the interior and annular domains. For solving the oscillatory Helmholtz equation in the exterior domain, the form (4.1) and, hence, the above algorithm are not appropriate due to the requirement of the Sommerfeld radiation condition at the far field as discussed in some detail in section 3.2.2. Later we discuss the algorithm for this case. For the monotonic Helmholtz equation, the above algorithm is also stable for all three different types of domains.

Depending on the particular problem we have to solve, we can reduce the amount of calculations by exploiting some relations between the integrals $Q_n(r)$ with $n > 0$ and those with $n < 0$. Except for the problem associated with the oscillatory Helmholtz operator in the exterior of a closed disc, $v_{n,1}(r)$ and $v_{n,2}(r)$ associated with all the other problems considered in the previous section are real functions. When the function f is real, i.e. $f_{-n}(r) = \overline{f_n(r)}$, using (3.24), (3.32) and (3.51) we obtain that the integrals in (4.1) satisfy

$$\begin{aligned} Q_{-n}^{(1)}(r_m) &= \overline{Q_n^{(1)}(r_m)}, \quad m = 2, \dots, M, \\ Q_{-n}^{(2)}(r_m) &= \overline{Q_n^{(2)}(r_m)}, \quad m = 1, \dots, M-1, \\ Q_{-n}(r_m) &= \overline{Q_n(r_m)}, \quad\ \ m = 1, \dots, M. \end{aligned} \tag{4.6}$$

Algorithm 4.1A together with the relations given in (4.6) provides values of Q_n for all the modes for all the problems except for the exterior oscillatory Helmholtz problem which we discuss below after the following remark.

Remark 4.1. For exterior problems with the support of f contained in the circle of radius R_0, $R_1 < R_0 < R_2$, the support of the Fourier coefficients f_n will be contained in the segment $[R_1, R_0]$, and consequently,

$$Q_n^{(1)}(r_m) = \frac{v_{n,2}(r_m)}{v_{n,2}(R_0)} Q_n^{(1)}(R_0) \quad \text{and} \quad Q_n^{(2)}(r_m) = 0, \quad \text{for any } R_0 < r_m \leqslant R_2. \tag{4.7}$$

Therefore, it follows that

$$Q_n(r_m) = Q_n^{(1)}(r_m), \quad \text{for any } R_0 < r_m \leqslant R_2. \tag{4.8}$$

Use of this in algorithm 4.1B will reduce the computational effort considerably when the support of f is small in comparison with the domain we consider for the problem.

For the problem associated with the oscillatory Helmholtz operator in the exterior of a closed disc, we need to compute the integrals $S_n(r)$ in (4.2). Algorithm 4.1B below is used to compute the values of the integrals $S_n(r)$ only for $n \geqslant 0$, and then using the relations (4.12) given below, we find the values of these integrals for $n < 0$.

Algorithm 4.1B – Sequential algorithm for the integrals S_n.
Step 1. Compute $S_n^{(1)}(r_m)$, $m = 2, \ldots, M$, using

$$\begin{aligned} S_n^{(1)}(r_2) &= \int_{r_1}^{r_2} \rho v_{n,1}(r_2) v_{n,2}(\rho) f_n(\rho)\,\mathrm{d}\rho, \\ S_n^{(1)}(r_m) &= \frac{v_{n,1}(r_m)}{v_{n,1}(r_{m-1})} S_n^{(1)}(r_{m-1}) + \int_{r_{m-1}}^{r_m} \rho v_{n,1}(r_m) v_{n,2}(\rho) f_n(\rho)\,\mathrm{d}\rho, \quad m = 3, \ldots, M. \end{aligned} \tag{4.9}$$

Step 2. Compute $S_n^{(2)}(r_m)$, $m = M-1, \ldots, 1$, using

$$\begin{aligned} S_n^{(2)}(r_{M-1}) &= \int_{r_{M-1}}^{r_M} \rho v_{n,2}(r_{M-1}) v_{n,1}(\rho) f_n(\rho)\,\mathrm{d}\rho, \\ S_n^{(2)}(r_m) &= \frac{v_{n,2}(r_m)}{v_{n,2}(r_{m+1})} S_n^{(2)}(r_{m+1}) + \int_{r_m}^{r_{m+1}} \rho v_{n,2}(r_m) v_{n,1}(\rho) f_n(\rho)\,\mathrm{d}\rho, \\ m &= M-2, \ldots, 1. \end{aligned} \tag{4.10}$$

Step 3. Compute $S_n(r_m)$, $m = 1, \ldots, M$, using

$$\begin{aligned} &S_n(r_1) = S_n^{(2)}(r_1), \qquad S_n(r_M) = S_n^{(1)}(r_M), \\ &S_n(r_m) = S_n^{(1)}(r_m) + S_n^{(2)}(r_m), \quad m = 2, \ldots, M-1. \end{aligned} \tag{4.11}$$

We obtain from (3.38) and (3.39) that $v_{n,1}(r) = H_n^{(1)}(kr) = \overline{H_n^{(2)}(kr)} = \overline{v_{n,2}(r)}$. Therefore, the following relations follow from (3.43) and the fact that f is a real function.

$$\begin{aligned}
S_{-n}^{(1)}(r_m) &= \overline{\frac{v_{n,2}(r_m)}{v_{n,2}(r_1)} S_n^{(2)}(r_1) - S_n^{(2)}(r_m)}, && m = 2, \ldots, M, \\
S_{-n}^{(2)}(r_m) &= \overline{\frac{v_{n,1}(r_m)}{v_{n,1}(r_M)} S_n^{(1)}(r_M) - S_n^{(1)}(r_m)}, && m = 2, \ldots, M, \\
S_{-n}(r_m) &= \overline{\frac{v_{n,2}(r_m)}{v_{n,2}(r_1)} S_n(r_1) + \frac{v_{n,1}(r_m)}{v_{n,1}(r_M)} S_n(r_M) - S_n(r_m)}, && m = 1, \ldots, M.
\end{aligned} \tag{4.12}$$

Algorithm 4.1B together with the relations given in (4.12) provide values of S_n rapidly for all the modes required for solving the exterior oscillatory Helmholtz problem. Also note the following remark, which can further speed up the computation.

Remark 4.2. If the support of f is contained in the circle of radius R_0, $R_1 < R_0 < R_2$, then the support of the Fourier coefficients f_n will be contained in the segment $[R_1, R_0]$, and consequently,

$$S_n^{(1)}(r_m) = \frac{v_{n,1}(r_m)}{v_{n,1}(R_0)} S_n^{(1)}(R_0) \quad \text{and} \quad S_n^{(2)}(r_m) = 0 \quad \text{for any } R_0 < r_m \leqslant R_2. \tag{4.13}$$

Therefore, it follows that

$$S_n(r_m) = S_n^{(1)}(r_m) \quad \text{for any } R_0 < r_m \leqslant R_2. \tag{4.14}$$

Use of this in algorithm 4.1B will reduce computational effort considerably when the support of f is small in comparison with the domain we consider for the problem.

For the numerical implementation, we recall the well-known recurrence relations for the Bessel and Hankel functions

$$\begin{aligned}
J_{n+1}(r) &= \frac{2n}{r} J_n(r) - J_{n-1}(r), & Y_{n+1}(r) &= \frac{2n}{r} Y_n(r) - Y_{n-1}(r), \\
H_{n+1}^{(1)}(r) &= \frac{2n}{r} H_n^{(1)}(r) - H_{n-1}^{(1)}(r), & H_{n+1}^{(2)}(r) &= \frac{2n}{r} H_n^{(2)}(r) - H_{n-1}^{(2)}(r), \\
I_{n+1}(r) &= -\tfrac{2n}{r} I_n(r) - I_{n-1}(r), & K_{n+1}(r) &= \frac{2n}{r} K_n(r) - K_{n-1}(r).
\end{aligned} \tag{4.15}$$

Consequently, the computation of the solutions of our problems using the above recurrence relations requires us to compute the values of the Bessel and Hankel functions by means of the series expansions only for $n = 0$ and $n = 1$. Then above recurrence relations can be used to compute the rest of the Bessel and Hankel functions which entails considerable computational saving provided the computation is carried out carefully. In this connection, it is worth making the following remarks. It may appear that this computational process is unstable because, in general, $|2n/r| > 1$. However, this would be so if the second term on the right-hand side of each of the recurrence relations in (4.15) were not present. Because of the presence of the second term in each of these, this instability does not occur unless $r \to 0$. For $r \to 0$, we do not use the above recurrence relation, rather we use the asymptotic formulae given in section 3, see (3.36), for example. We have been satisfied with the accuracy this computational process provides by comparing with direct calculations using series expansions of these functions.

4.2. *Algorithm for the two-dimensional problems*

Below we choose N an integer power of 2 for use in the Fourier series to calculate the solution $u(r, \theta)$ using the truncated version of (2.9). Now, we construct the following fast algorithm for various two-dimensional problems based on algorithms 4.1A and 4.1B and the analyses of the previous sections.

Algorithm 4.2 – Fast algorithm for the solution $u(r, \theta)$.
Initialization. Choose M and N. Define $K = N/2$.

Step 1. Using the fast Fourier transform, compute the Fourier coefficients $f_n(r_m)$, $-K \leqslant n \leqslant K - 1$ for $1 \leqslant m \leqslant M$.

Step 2. For the exterior oscillatory Helmholtz problem, compute the integrals $S_n(r_m)$, $-K \leqslant n \leqslant K - 1$, $1 \leqslant m \leqslant M$, using algorithm 4.1B for $n \geqslant 0$ and (4.12) for $n < 0$. For all other problems discussed in section 3, compute the integrals $Q_n(r_m)$, $-K \leqslant n \leqslant K - 1$, $1 \leqslant m \leqslant M$, using algorithm 4.1A for $n \geqslant 0$ and (4.6) for $n < 0$.

Step 3. Compute the values $v_n(r_m)$, $-K \leqslant n \leqslant K-1$ and $1 \leqslant m \leqslant M$, of the solutions of inhomogeneous problems (3.1) by multiplying $Q_n(r_m)$ ($S_n(r_m)$ for exterior oscillatory Helmholtz problem) with a problem dependent constant. The value of the constant is taken from one of the corollaries of propositions 3.2, 3.3 or 3.4 depending on the particular problem being solved.

Step 4. Compute the values of the Fourier coefficients of the solution $u(r, \theta)$, i.e. $u_n(r_m)$, $-K \leqslant n \leqslant K-1$ and $2 \leqslant m \leqslant M-1$, using the corollaries of propositions 3.2, 3.3 or 3.4, depending on the particular problem being solved.

Step 5. Finally, compute the values $u(r_m, \theta_n)$, $-K \leqslant n \leqslant K - 1$ and $2 \leqslant m \leqslant M - 1$, of the solution $u(r, \theta)$ using the fast Fourier transform.

4.3. *The algorithmic complexity*

In steps 1 and 5 above, there are 2M FFT's of length N and all other computations in steps 2, 3, and 4 are of lower order. With each FFT of length N contributing $N \log N$ operations, the asymptotic operation count and hence the asymptotic time complexity is

$O(MN \log N)$. It is easy to see that the asymptotic storage requirement is of the order $O(MN)$.

Finally, we remark that by construction the above algorithm 4.2 is parallelizable on multi-processor as discussed in detail for two similar algorithms in [5–7].

5. Numerical results

In this section, numerical results are presented along with the accuracy of the above algorithm applied to the problems discussed earlier. In all the examples presented below, we have considered problems with known solutions and compared the numerical results with them. For each of the three operators, we have considered both the Dirichlet and the Neumann problems. In the case of an annular domain, we have also considered a problem with mixed boundary conditions.

The algorithm is applied with equidistant points along the radius, $R_1 = r_1 < r_2 < \cdots < r_M = R_2$. The distance between two consecutive radial points has been denoted by δ_r. The two integrals in (4.3) and (4.4) of algorithm 4.1A have been approximated by the trapezoidal rule on the segments $[r_{m-1}, r_m]$ and $[r_m, r_{m+1}]$, respectively. The total number of the Fourier coefficients used has been denoted, as before, by N.

In the examples we consider the total variation in the values of the solution is large and the minimum value of the solution in the domain is of the order of round-off error. Therefore, it is more appropriate to calculate the error relative to the maximum value of the exact solution in the domain. Denoting, as in the previous sections, by u the exact solution, and by u_c the computed solution, we list in the tables below the maximum of the relative error,

$$\mathrm{err} = \max_{2\leqslant i\leqslant M-1,\ 1\leqslant j\leqslant N} \frac{|u(r_i,\theta_j) - u_c(r_i,\theta_j)|}{C_0} \quad \text{where } C_0 = \max_{(x,y)\in\overline{\Omega}} \big|u(x,y)\big|, \tag{5.1}$$

over the $N(M-2)$ points in the domain.

Interior circular domains. Here we have considered problems with the Helmholtz equation

$$\Delta u - k^2 u = f, \tag{5.2}$$

where f and the boundary conditions have been chosen such that the problem has the solution

$$u(x,y) = x\mathrm{e}^x + y\mathrm{e}^y. \tag{5.3}$$

The domain of the problems is the disc centered at the origin and of radius $R = 1$, and the maximum value of the exact solution $u(x,y)$ on the closed disc is $C_0 = u(1/\sqrt{2}, 1/\sqrt{2}) \approx 0.287\mathrm{E}{+}01$. In the tables below, we show the relative error (5.1) for regular meshes of the disc with δ_r varying between 1.0/8 and 1.0/8192, and N between 4 and 64. Tables 1 and 2 show this error for the Dirichlet and Neumann problems, respectively, where the constant in (5.2) is $k^2 = 1.0$.

Table 1
Errors for the Dirichlet problem in a disc when $k^2 = 1.0$.

$\delta_r \backslash N$	4	8	16	32	64
2^{-3}	0.294E−01	0.338E−02	0.337E−02	0.337E−02	0.337E−02
2^{-4}	0.287E−01	0.116E−02	0.115E−02	0.115E−02	0.115E−02
2^{-5}	0.286E−01	0.368E−03	0.367E−03	0.367E−03	0.367E−03
2^{-6}	0.286E−01	0.242E−03	0.111E−03	0.111E−03	0.111E−03
2^{-7}	0.286E−01	0.238E−03	0.328E−04	0.328E−04	0.328E−04
2^{-8}	0.286E−01	0.237E−03	0.942E−05	0.942E−05	0.942E−05
2^{-9}	0.285E−01	0.237E−03	0.266E−05	0.266E−05	0.266E−05
2^{-10}	0.285E−01	0.237E−03	0.743E−06	0.743E−06	0.743E−06
2^{-11}	0.285E−01	0.237E−03	0.205E−06	0.205E−06	0.205E−06
2^{-12}	0.285E−01	0.237E−03	0.863E−07	0.678E−07	0.656E−07
2^{-13}	0.285E−01	0.237E−03	0.863E−07	0.679E−07	0.657E−07

Table 2
Errors for the Neumann problem in a disc when $k^2 = 1.0$.

$\delta_r \backslash N$	4	8	16	32	64
2^{-3}	0.191E+00	0.101E−01	0.936E−02	0.936E−02	0.936E−02
2^{-4}	0.199E+00	0.323E−02	0.248E−02	0.248E−02	0.248E−02
2^{-5}	0.202E+00	0.141E−02	0.636E−03	0.636E−03	0.636E−03
2^{-6}	0.202E+00	0.939E−03	0.161E−03	0.161E−03	0.161E−03
2^{-7}	0.202E+00	0.819E−03	0.406E−04	0.406E−04	0.406E−04
2^{-8}	0.202E+00	0.815E−03	0.102E−04	0.102E−04	0.102E−04
2^{-9}	0.202E+00	0.817E−03	0.258E−05	0.256E−05	0.257E−05
2^{-10}	0.202E+00	0.817E−03	0.703E−06	0.667E−06	0.666E−06
2^{-11}	0.202E+00	0.817E−03	0.234E−06	0.198E−06	0.197E−06
2^{-12}	0.202E+00	0.817E−03	0.117E−06	0.834E−07	0.797E−07
2^{-13}	0.202E+00	0.817E−03	0.876E−07	0.549E−07	0.508E−07

We see in these two tables that the errors are approximately of the same order for the two types of boundary conditions of the problem. This fact has also been noticed with other values of the parameter k. For this reason, in table 3 only the error for the Dirichlet problem when $k^2 = 0.5$ is shown. Also, in table 4 the accuracy of the algorithm is illustrated only for the Neumann problem when $k^2 = 5.0$.

It is worth making three remarks. Firstly, although the trapezoidal rule is used to approximate the integrals in algorithm 4.1A, which is accurate of the order $O(\delta_r^2)$, the numerical solutions are accurate up to five decimal places when $\delta_r = 10^{-3}$ and $N = 16$. It is expected that the use of a three-point based integration method, such as Simpson's rule, which is accurate of the order $O(\delta_r^4)$, may provide even more accurate solutions with the same number of nodes.

Secondly, the above tables show that both the number of nodes along the radius and the number of Fourier coefficients used are important for the accuracy of the algorithm.

Table 3
Errors for the Dirichlet problem in a disc when $k^2 = 0.5$.

$\delta_r \backslash N$	4	8	16	32	64
2^{-3}	0.326E−01	0.438E−02	0.438E−02	0.438E−02	0.438E−02
2^{-4}	0.312E−01	0.141E−02	0.141E−02	0.141E−02	0.141E−02
2^{-5}	0.309E−01	0.431E−03	0.430E−03	0.430E−03	0.430E−03
2^{-6}	0.308E−01	0.251E−03	0.127E−03	0.127E−03	0.127E−03
2^{-7}	0.308E−01	0.243E−03	0.367E−04	0.367E−04	0.367E−04
2^{-8}	0.308E−01	0.241E−03	0.104E−04	0.104E−04	0.104E−04
2^{-9}	0.308E−01	0.241E−03	0.291E−05	0.291E−05	0.291E−05
2^{-10}	0.308E−01	0.241E−03	0.803E−06	0.803E−06	0.803E−06
2^{-11}	0.308E−01	0.241E−03	0.220E−06	0.220E−06	0.220E−06
2^{-12}	0.308E−01	0.241E−03	0.863E−07	0.678E−07	0.656E−07
2^{-13}	0.308E−01	0.241E−03	0.863E−07	0.679E−07	0.657E−07

Table 4
Errors for the Neumann problem in a disc when $k^2 = 5.0$.

$\delta_r \backslash N$	4	8	16	32	64
2^{-3}	0.482E−01	0.312E−01	0.316E−01	0.316E−01	0.316E−01
2^{-4}	0.282E−01	0.932E−02	0.957E−02	0.957E−02	0.957E−02
2^{-5}	0.218E−01	0.278E−02	0.263E−02	0.263E−02	0.263E−02
2^{-6}	0.200E−01	0.941E−03	0.688E−03	0.688E−03	0.689E−03
2^{-7}	0.195E−01	0.452E−03	0.176E−03	0.176E−03	0.176E−03
2^{-8}	0.194E−01	0.370E−03	0.445E−04	0.445E−04	0.446E−04
2^{-9}	0.194E−01	0.378E−03	0.112E−04	0.112E−04	0.112E−04
2^{-10}	0.194E−01	0.380E−03	0.277E−05	0.280E−05	0.281E−05
2^{-11}	0.194E−01	0.380E−03	0.683E−06	0.700E−06	0.702E−06
2^{-12}	0.194E−01	0.381E−03	0.179E−06	0.176E−06	0.174E−06
2^{-13}	0.194E−01	0.381E−03	0.752E−07	0.496E−07	0.455E−07

Examining the rows of the tables, we observe that for a given number of points along the radius, increasing the number of Fourier coefficients decreases the error until a certain value is reached and thereafter, it remains the same. Examining the columns, the same kind of dependence of the error on the number of nodes along the radius, for a fixed number of Fourier coefficients, is observed.

Thirdly, one should expect order δ_r^2 convergence at least in the middle of the columns of these tables for large N. In tables 2 and 4 this seems to be the case for $N = 64$, in tables 1 and 4 it is strictly not, and there is a slight deterioration from this convergence. We do not know the reason, but perhaps higher values of N are required to observe this convergence rate.

Annular domains. Here we have considered solving the Poisson equation

$$\Delta u = f \tag{5.4}$$

Table 5
Errors for the Dirichlet problem in an annulus.

$\delta_r \backslash N$	4	8	16	32	64
2^{-2}	0.566E−01	0.747E−02	0.182E−02	0.180E−02	0.181E−02
2^{-3}	0.555E−01	0.609E−02	0.470E−03	0.457E−03	0.458E−03
2^{-4}	0.551E−01	0.574E−02	0.129E−03	0.114E−03	0.115E−03
2^{-5}	0.550E−01	0.566E−02	0.446E−04	0.286E−04	0.287E−04
2^{-6}	0.549E−01	0.563E−02	0.248E−04	0.715E−05	0.717E−05
2^{-7}	0.549E−01	0.563E−02	0.202E−04	0.179E−05	0.179E−05
2^{-8}	0.549E−01	0.563E−02	0.202E−04	0.446E−06	0.447E−06
2^{-9}	0.549E−01	0.563E−02	0.202E−04	0.111E−06	0.111E−06
2^{-10}	0.549E−01	0.563E−02	0.202E−04	0.578E−07	0.612E−07
2^{-11}	0.549E−01	0.563E−02	0.202E−04	0.579E−07	0.613E−07
2^{-12}	0.549E−01	0.563E−02	0.202E−04	0.579E−07	0.614E−07

Table 6
Errors for the mixed problem in an annulus.

$\delta_r \backslash N$	4	8	16	32	64
2^{-2}	0.333E+00	0.202E−01	0.255E−01	0.255E−01	0.255E−01
2^{-3}	0.362E+00	0.243E−01	0.690E−02	0.697E−02	0.697E−02
2^{-4}	0.369E+00	0.297E−01	0.175E−02	0.182E−02	0.182E−02
2^{-5}	0.371E+00	0.312E−01	0.456E−03	0.464E−03	0.464E−03
2^{-6}	0.372E+00	0.315E−01	0.161E−03	0.117E−03	0.117E−03
2^{-7}	0.372E+00	0.316E−01	0.869E−04	0.295E−04	0.295E−04
2^{-8}	0.372E+00	0.316E−01	0.682E−04	0.739E−05	0.739E−05
2^{-9}	0.372E+00	0.317E−01	0.637E−04	0.185E−05	0.188E−05
2^{-10}	0.372E+00	0.317E−01	0.649E−04	0.489E−06	0.503E−06
2^{-11}	0.372E+00	0.317E−01	0.652E−04	0.155E−06	0.158E−06
2^{-12}	0.372E+00	0.317E−01	0.653E−04	0.713E−07	0.716E−07

in an annular domain bounded by the radii $R_1 = 2.0$ and $R_2 = 4.0$ with f and the boundary conditions appropriately chosen such that the problem has the solution

$$u(x, y) = x\mathrm{e}^{x} - y\mathrm{e}^{-y}. \tag{5.5}$$

The maximum value of the exact solution $u(x, y)$ in the closed annulus, depending on which we have calculated the relative errors for the numerical solution, is $C_0 = u(4.0, 0.0) \approx 0.218\mathrm{E}{+}03$. Similar to the previous examples, we show in tables 5–7 the relative error (5.1) for regular meshes of the annulus with δ_r varying between $1.0/4$ and $1.0/4096$, and N between 4 and 64. Table 5 gives the error for the problem with Dirichlet data on both circles of the boundary. For table 6 we have taken Dirichlet data on $r = R_1$ and Neumann data on $r = R_2$. In the numerical experiments for problems with Neumann data on $r = R_1$ and Dirichlet data on $r = R_2$, the errors have been found

Table 7
Errors for the Neumann problem in an annulus.

$\delta_r \backslash N$	4	8	16	32	64
2^{-2}	0.929E+00	0.432E−01	0.426E−01	0.426E−01	0.426E−01
2^{-3}	0.967E+00	0.681E−01	0.105E−01	0.105E−01	0.105E−01
2^{-4}	0.977E+00	0.738E−01	0.262E−02	0.262E−02	0.262E−02
2^{-5}	0.979E+00	0.752E−01	0.647E−03	0.655E−03	0.655E−03
2^{-6}	0.980E+00	0.754E−01	0.196E−03	0.164E−03	0.164E−03
2^{-7}	0.980E+00	0.755E−01	0.918E−04	0.409E−04	0.409E−04
2^{-8}	0.980E+00	0.755E−01	0.729E−04	0.102E−04	0.102E−04
2^{-9}	0.980E+00	0.755E−01	0.682E−04	0.256E−05	0.256E−05
2^{-10}	0.980E+00	0.755E−01	0.680E−04	0.640E−06	0.640E−06
2^{-11}	0.980E+00	0.755E−01	0.684E−04	0.164E−06	0.161E−06
2^{-12}	0.980E+00	0.755E−01	0.685E−04	0.692E−07	0.686E−07

to be of the same order as that in table 6. Finally, in table 7 the error for the problem with Neumann data on both circles are shown.

The three tables show that the errors in the case of an annulus are similar to those obtained within a disc. Similar remarks concerning the accuracy of the method and the dependence of the error on the number of radial points or the number of the terms in the Fourier expansion can be made.

Exterior circular domains. Here we have considered solving the Helmholtz equation

$$\Delta u + k^2 u = f, \tag{5.6}$$

in the exterior of a circle of radius R_1. In the general theory in section 2, the function f has compact support in the case of exterior problems. However, the existence and uniqueness of the solution are guaranteed for inhomogeneous equations with f decaying rapidly to zero as $r \to \infty$. The algorithm is applied to solve the exterior problems with f and the boundary data such that the problem has the solution

$$u(x, y) = \mathrm{e}^{-x^2-y^2+x+y}. \tag{5.7}$$

Using (5.7) for $u(x, y)$ in the Helmholtz equation (5.6), we find that

$$f(x, y) = \left[4\left(x^2 + y^2 - x - y\right) - 2 + k^2\right]\mathrm{e}^{-x^2-y^2+x+y}.$$

For points with $r > 7.0$, the values of f are less than 0.153E−14 when $k^2 = 5.0$. Also, we notice that $u(x, y)$ in (5.7) satisfies the conditions (3.79) at infinity and its values at points $r > 7.0$ are less than 0.104E−16. Consequently, we have chosen $R_2 = 7.0$. Also we have taken $R_1 = 1.0$, and we see that the maximum value of the exact solution $u(x, y)$ is $C_0 = u(1/\sqrt{2}, 1/\sqrt{2}) \approx 0.151\text{E}{+}01$. In tables 8 and 9 we give the errors we have obtained for the Dirichlet and Neumann problems taking $k^2 = 5.0$.

We notice that the results we have obtained for the exterior problems and the bounded domains are similar.

Table 8
Errors for the exterior Dirichlet problem when $k^2 = 5.0$.

$\delta_r \backslash N$	4	8	16	32
$3.0/2^3$	0.329E+00	0.523E+00	0.520E+00	0.520E+00
$3.0/2^4$	0.613E−01	0.133E+00	0.131E+00	0.131E+00
$3.0/2^5$	0.109E−01	0.349E−01	0.324E−01	0.324E−01
$3.0/2^6$	0.204E−01	0.107E−01	0.812E−02	0.812E−02
$3.0/2^7$	0.245E−01	0.461E−02	0.203E−02	0.203E−02
$3.0/2^8$	0.255E−01	0.309E−02	0.509E−03	0.509E−03
$3.0/2^9$	0.257E−01	0.271E−02	0.129E−03	0.127E−03
$3.0/2^{10}$	0.258E−01	0.262E−02	0.339E−04	0.317E−04
$3.0/2^{11}$	0.258E−01	0.260E−02	0.102E−04	0.793E−05
$3.0/2^{12}$	0.258E−01	0.259E−02	0.485E−05	0.199E−05

Table 9
Errors for the exterior Neumann problem when $k^2 = 5.0$.

$\delta_r \backslash N$	4	8	16	32
$3.0/2^3$	0.179E+00	0.342E+00	0.341E+00	0.341E+00
$3.0/2^4$	0.393E−01	0.942E−01	0.925E−01	0.925E−01
$3.0/2^5$	0.915E−02	0.252E−01	0.235E−01	0.235E−01
$3.0/2^6$	0.166E−01	0.762E−02	0.587E−02	0.587E−02
$3.0/2^7$	0.196E−01	0.323E−02	0.147E−02	0.147E−02
$3.0/2^8$	0.204E−01	0.215E−02	0.368E−03	0.367E−03
$3.0/2^9$	0.206E−01	0.188E−02	0.923E−04	0.918E−04
$3.0/2^{10}$	0.207E−01	0.181E−02	0.236E−04	0.230E−04
$3.0/2^{11}$	0.207E−01	0.180E−02	0.642E−05	0.579E−05
$3.0/2^{12}$	0.207E−01	0.179E−02	0.459E−05	0.149E−05

6. Conclusions

In this paper, analysis-based high-order accurate fast algorithms for solving elliptic problems in three different two-dimensional domains are presented and implemented: (i) interior of a circle, (ii) exterior of a circle, and (iii) circular annulus. These algorithms are derived from an exact formulae for the solution of a large class of elliptic equations (where the coefficients of the equation do not depend on the angle when we use the polar coordinates) based on Fourier series expansion and one-dimensional ordinary differential equation. In order to illustrate the application of these algorithms, three different types of elliptic problems considered are: (i) Poisson equation, (ii) Helmholtz equation (oscillatory case), and (iii) Helmholtz equation (monotone case). Numerical results are presented which exhibit the high accuracy of the proposed algorithms.

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References

[1] A. Averbuch, M. Israeli and L. Vozovoi, A fast Poisson solver of arbitrary order accuracy in rectangular regions, SIAM J. Sci. Comput. 19 (1998) 933–952.
[2] L. Badea and P. Daripa, On fast direct elliptic solver by modified Fourier method, Numer. Algorithms 15 (1997) 287–313.
[3] L. Badea and P. Daripa, On a domain embedding method using the optimal distributed control and a fast algorithm, submitted.
[4] B. Bialecki, A fast domain decomposition Poisson solver on a rectangle for Hermite bicubic orthogonal spline collocation, SIAM J. Numer. Anal. 30 (1993) 425–434.
[5] L. Borges and P. Daripa, A parallel solver for singular integrals, in: *Proc. of PDPTA'99 – Internat. Conf. on Parallel and Distributed Processing Techniques and Applications*, Vol. III, Las Vegas, Nevada, 28 June–1 July 1999, pp. 1495–1501.
[6] L. Borges and P. Daripa, A parallel version of a fast algorithm for singular integral transforms, Numer. Algorithms 23 (2000) 71–96.
[7] L. Borges and P. Daripa, A fast parallel algorithms for the Poisson equation on a disk, J. Comput. Phys. 169 (2001) 151–192.
[8] E. Braverman, M. Israeli, A. Averbuch and L. Vozovoi, A fast 3D Poisson solver of arbitrary order accuracy, J. Comput. Phys. 144 (1998) 109–136.
[9] T.F. Chan and D.C. Resasco, A domain-decomposed fast Poisson solver on a rectangle, SIAM J. Sci. Statist. Comput. 8 (1987) S14–S26.
[10] T.F. Chan and F. Saied, A comparison of elliptic solvers for general two-dimensional regions, SIAM J. Sci. Statist. Comput. 6 (1985) 742–760.
[11] S.C. Chang, Solution of elliptic PDEs by fast Poisson solvers using a local relaxation factor, J. Comput. Phys. 67 (1986) 91–123.
[12] P. Daripa, On applications of a complex variable method in compressible flows, J. Comput. Phys. 88 (1990) 337–361.
[13] P. Daripa, A fast algorithm to solve nonhomogeneous Cauchy–Riemann equations in the complex plane, SIAM J. Sci. Statist. Comput. 13 (1992) 1418–1432.
[14] P. Daripa, A fast algorithm to solve the Beltrami equation with applications to quasiconformal mappings, J. Comput. Phys. 106 (1993) 355–365.
[15] P. Daripa and D. Mashat, Singular integral transforms and fast numerical algorithms, Numer. Algorithms 18 (1998) 133–157.
[16] P. Daripa and D. Mashat, An efficient numerical method for quasiconformal mappings of doubly connected domains, Numer. Algorithms 18 (1998) 159–178.
[17] R. Dautray and J.L. Lions, eds., *Analyse Mathématique et Calcul Numérique, I, Modèles Physiques* (Masson, Paris, 1987).
[18] M.D. Greenberg, *Application of Green's Functions in Science and Engineering* (Prentice-Hall, Englewood Cliffs, NJ, 1971).
[19] L. Greengard and J.Y. Lee, A direct adaptive Poisson solver of arbitrary order accuracy, J. Comput. Phys. 125 (1996) 415–424.

[20] L. Greengard and V. Rokhlin, A new version of the fast multipole method for the Laplace equation in three dimensions, Acta Numerica (1997) 229–269.
[21] R.W. Hockney, A fast direct solution of Poisson's equation using Fourier analysis, J. Assoc. Comput. Math. 12 (1965) 95–113.
[22] E. Houstis, R. Lynch and J. Rice, Evaluation of numerical methods for ellipitic partial differential equations, J. Comput. Phys. 27 (1978) 323–350.
[23] A. McKenney, L. Greengard and A. Mayo, A fast Poisson solver for complex geometries, J. Comput. Phys. 118 (1995) 348–355.
[24] D. O'Leary, and O. Widlund, Capacitance, matrix methods for the Helmholtz equation on general three-dimensional regions, Math. Comp. 33 (1979) 849–879.
[25] A. Onana, S.V. Kwankam and E. Zoue, A fast Poisson solver, Period. Math. Hungar. 28 (1994) 89–101.
[26] L.A. Pipes and L.R. Harvill, *Applied Mathematics for Engineers and Physicists*, 3rd ed. (McGraw-Hill, New York, 1970).
[27] J. Rice, E. Houstis and R. Dyksen, A population of linear, second order, elliptic partial differential equations on rectangular domains. I, II, Math. Comp. 36 (1981) 475–484.
[28] J. Shen, Efficient Galerkin methods III: Polar and cylindrical geometries, SIAM J. Sci. Comput. 18 (1997) 1583–1604.
[29] A.S.L. Shieh, Fast Poisson solvers on general two-dimensional regions for the Dirichlet problem, Numer. Math. 31 (1978) 405–429.
[30] G. Sköllermo, A Fourier method for the numerical solution of Poisson's equation, Math. Comp. 29 (1975) 697–711.
[31] P.N. Swarztrauber, The methods of cyclic reduction, Fourier analysis and the FACR algorithm for the discrete solution of Poisson's equation on a rectangle, SIAM Rev. 19 (1977) 491–501.
[32] P.N. Swarztrauber and R.A. Sweet, A direct method for the discrete Poisson equation on a disc, SIAM J. Numer. Anal. 10 (1973) 900–907.
[33] C. Temperton, On the FACR(ℓ) algorithm for the discrete Poisson equation, J. Comput. Phys. 34 (1980) 315–329.
[34] K. Yosida, *Lectures on Differential and Integral Equations* (Dover, New York, 1991).