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On a Fourier method of embedding domains using an optimal distributed control

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We propose a domain embedding method to solve second order elliptic problems in arbitrary two-dimensional domains. This method can be easily extended to three-dimensional problems. The method is based on formulating the problem as an optimal distributed control problem inside a rectangle in which the arbitrary domain is embedded. A periodic solution of the equation under consideration is constructed easily by making use of Fourier series. Numerical results obtained for Dirichlet problems are presented. The numerical tests show a high accuracy of the proposed algorithm and the computed solutions are in very good agreement with the exact solutions.

Keywords: domain embedding methods, optimal distributed control, spectral methods

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1. Introduction

In embedding or fictitious domain methods, complicated domains Ω, where solutions of problems may be sought, are embedded into larger domains D with simple enough boundaries (circles, rectangles, for instance) so that the solutions in these domains can be constructed by efficient methods. For instance, there are many problems for which either exact solutions on discs or rectangles are known or accurate numerical solutions on regular domains can be constructed using fast solvers. Also, for rectangular domains we can use a regular mesh associated with a spectral method or a finite element method. The solution of the problem in Ω is then approximated by a solution of the equation in fictitious domain D satisfying the boundary conditions of the given problem in Ω approximately, if not exactly. The proposed embedding method below is of real interest since it works well for complex geometries unlike some other methods, especially the finite difference method.

These methods were developed in the seventies [1,6,12,22,24,25] and have been a very active area of research in the following decades. The use of these embedding methods is commonplace these days for solving complicated problems arising in science and engineering. To this end, it is worth mentioning the domain embedding methods for Stokes equations [4], fluid dynamics and electro-magnetics [9], transonic flow calculation [26], and equilibrium of the plasma in a Tokamak [3]. Also, there has been an enormous progress in shape optimization using the fictitious domain approaches. We can cite here, for instance, the works of Daňková, Haslinger, Klarbring, Makinen, Neittaanmäki and Tiba (see [7,18–20,23]) among many others. In recent years, progress in this field has been substantial, especially in the use of the Lagrange multiplier techniques. In this connection, the works of Girault, Glowinski, Hesla, Joseph, Kuznetsov, Lopez, Pan, Périaux [13–17] should be cited.

In [2], an embedding method is associated with an optimal distributed control problem (see [21]). There, the problem is solved in an auxiliary domain D using a finite element method on a fairly structured mesh which allows the use of fast solvers. The solution in D is found as a solution of an optimal distributed control problem such that it satisfies the prescribed boundary conditions of the problem in the domain Ω. The same idea is also used in [8] where a least squares method is used. An embedding method is proposed in [10,11], where a combination of Fourier approximations and boundary integral equations is used. Essentially, a Fourier approximation for a solution of the nonhomogeneous equation in D is found, and then the solution in Ω for the homogeneous equation subject to modified conditions on the boundary of Ω is sought using boundary element methods. The embedding method proposed in this paper uses the Fourier spectral method proposed in [10] to find a periodic solution of the equation associated with the problem in D and, as in [2], the approximate solution of the problem in Ω is found as the solution of an optimal distributed control problem. To this end, it is worth mentioning the works of Briscolini and Santangelo [5] and Elghaoui and Pasquetti [11].

The paper is organized as follows. In section 2 we describe the proposed method and apply it to solve some elliptic problems in complicated domains. In section 3 we present some numerical results which show the validity and high accuracy of the method. Finally, we provide some concluding remarks in section 4.

2. Description of the method

In numerically time-advancing solutions of unsteady advection–diffusion problems (see [10]), one often requires to solve at each time step one or more linear elliptic problems of the type

$$\begin{aligned} &\Delta u - \sigma u = f \quad && \text{in } \Omega, \\ &u = g && \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where σ is a positive constant and Γ is the boundary of the domain $\Omega \subset \mathbb{R}^2$. In [10], the domain Ω is embedded in a rectangle $D = (0, a) \times (0, b)$ on which a regular rectangular mesh is considered. The proposed algorithm in [10] has four steps. In the first one, the function f is extended from the domain Ω to a function $\tilde{f}$ defined on D, $\tilde{f}_{|\Omega} = f$. Using the mesh on D, a discrete Fourier expansion of a periodic solution $\tilde{u}$ of the equation

$$\Delta\tilde{u} - \sigma\tilde{u} = \tilde{f} \quad \text{in } D \tag{2.2}$$

is written in the second step. In the third step, the problem

$$\begin{aligned} &\Delta v - \sigma v = 0 \quad \text{in } \Omega,\\ &u = g - \tilde{u}_{|\Gamma} \quad \text{on } \Gamma. \end{aligned} \tag{2.3}$$

is solved using the boundary element method. Finally, the desired solution $u = \tilde{u} + v$ is calculated on Ω in the fourth step.

It is worth making some remarks on this algorithm before we propose a new algorithm in which the same spectral method for the solution of problem (2.2) is used, but now the solution of problem (2.1) is found by an optimal distributed control.

In the method proposed above, the periodic solution $\tilde{u}$ of equation (2.2) is used to reduce the problem (2.1) for an inhomogeneous equation to a problem (2.3) for a homogeneous equation, thereby allowing the use of boundary element method to solve the problem (2.3).

If we know $\tilde{f}$ and write it as a double Fourier series

$$\tilde{f}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{f}_{mn} \mathrm{e}^{\mathrm{i}(mX+nY)}, \tag{2.4}$$

where $X = 2\pi x/a$, $Y = 2\pi y/b$, then a periodic solution $\tilde{u}$ of (2.2) can be found from its Fourier series,

$$\tilde{u}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{u}_{mn} \mathrm{e}^{\mathrm{i}(mX+nY)}, \tag{2.5}$$

with $\hat{u}_{mn}$ now given by

$$\hat{u}_{mn} = -\frac{\hat{f}_{mn}}{((4\pi^2/a^2)m^2 + (4\pi^2/b^2)n^2 + \sigma)}. \tag{2.6}$$

If we consider a rectangular mesh on $[0, a] \times [0, b]$ with $2M + 1$ and $2N + 1$ segments on $[0, a]$ and $[0, b]$, respectively, then the discrete Fourier transforms approximating $\tilde{f}$ and $\tilde{u}$ are given by

$$\tilde{f}_{ij} = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{f}_{mn} \mathrm{e}^{\mathrm{i}(mX_i+nY_j)} \tag{2.7}$$

and

$$\tilde{u}_{ij} = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{u}_{mn} \mathrm{e}^{\mathrm{i}(mX_i+nY_j)}, \tag{2.8}$$

respectively, where $\tilde{f}_{ij} = \tilde{f}(x_i, y_j)$, $\tilde{u}_{ij} = \tilde{u}(x_i, y_j)$, $X_i = 2\pi x_i/a$, and $Y_j = 2\pi y_j/b$. It is well known that the inverse discrete Fourier transforms of (2.7) and (2.8) are written as

$$\hat{f}_{mn} = \frac{1}{(2M+1)(2N+1)} \sum_{i=0}^{2M} \sum_{j=0}^{2N} \tilde{f}_{ij} \mathrm{e}^{-\mathrm{i}(mX_i + nY_j)} \tag{2.9}$$

and

$$\hat{u}_{mn} = \frac{1}{(2M+1)(2N+1)} \sum_{i=0}^{2M} \sum_{j=0}^{2N} \tilde{u}_{ij} \mathrm{e}^{-\mathrm{i}(mX_i + nY_j)}, \tag{2.10}$$

respectively. Therefore, for arbitrary values of $\tilde{f}$ at the mesh nodes in $\overline{D}\backslash\overline{\Omega}$, we can find a Fourier approximation of the periodic solution of equation (2.2) using (2.9), (2.6) and (2.8). However, in this approach there can be large variations in the values of $\tilde{f}$ at the mesh nodes in $\overline{D}\backslash\overline{\Omega}$ which may cause large errors in the boundary conditions of the problem (2.3) for some $\tilde{f}$. We point out here that the discretization points of the boundary Γ do not coincide in general with the nodes of the regular mesh on D. In order to avoid a stiff gradient of $\tilde{f}$, this one is calculated in [10] as the solution of the following optimization problem

$$\left|\Delta^p \tilde{f}\right|_{L^2_{\mathrm{per}}(D)} = \min_{\tilde{h}|\Omega = f} \left|\Delta^p \tilde{h}\right|_{L^2_{\mathrm{per}}(D)}, \tag{2.11}$$

where p is a fixed natural number. This minimization problem leads to solving an algebraic linear system for the unknowns $\tilde{f}_{ij}$, the values corresponding to the mesh nodes in $\overline{D}\backslash\overline{\Omega}$.

We find that if we obtain directly an extension of $\tilde{f}$ such that the trace on Γ of $\tilde{u}$ written in (2.8) is a good approximation for the boundary data g, then $\tilde{u}$ is also a good approximation in Ω for the solution u of problem (2.1). Construction of such a $\tilde{u}$ leads to an optimization problem too, but this time the values $\tilde{f}_{ij}$ at the mesh nodes in $\overline{D}\backslash\overline{\Omega}$ are sought such that g is well approximated at the discretization points on Γ. To be more explicit, we solve for $\tilde{f}$ the following optimal distributed control problem

$$\min_{\tilde{h}|\Omega = f} J(\tilde{h}), \quad J(\tilde{h}) = \frac{1}{2}\left|\tilde{u}(\tilde{h}) - g\right|^2_{L^2(\Gamma)}, \tag{2.12}$$

where $\tilde{u}(\tilde{h})$ is the periodic solution of problem (2.2) corresponding to the right-hand side $\tilde{h}$. In this way, the condition

$$\tilde{u} = g \quad \text{on } \Gamma$$

is satisfied approximately by $\tilde{u}(\tilde{f})$. Since J depends only on the values of $\tilde{h}$ at the mesh nodes in $\overline{D}\backslash\overline{\Omega}$, and if we write

$$\tilde{h} = f + h,$$

where f and h are assumed to be extended with zero in $\overline{D}\backslash\overline{\Omega}$ and $\overline{\Omega}$, respectively, then problem (2.12) can be written as

$$\min_h J(h), \quad J(h) = \frac{1}{2}\left|\tilde{u}(f+h) - g\right|^2_{L^2(\Gamma)}. \tag{2.13}$$

Since the Gâteaux derivative of $J(h)$ is

$$J'(h)(e) = \int_\Gamma \left[\tilde{u}(h) + \tilde{u}(f) - g\right]\tilde{u}(e),$$

the minimization problem (2.13) is equivalent with the linear algebraic system

$$\int_\Gamma \tilde{u}(h)\tilde{u}(e) = \int_\Gamma \left[g - \tilde{u}(f)\right]\tilde{u}(e), \tag{2.14}$$

for all e vanishing at the mesh nodes in $\overline{\Omega}$. The algebraic system (2.14) and that corresponding to (2.11) have the same dimension, but this time the spectral approximation of the solution u of problem (2.1) is the very solution obtained directly by solving algebraic system (2.14). Our numerical results show that the method proposed here is very efficient and has a high accuracy.

In order to write the matrix and the right hand side of the linear system (2.14), we denote by e_{ij} the discrete Fourier transform of the function which takes the value 1 at the mesh node (x_i, y_j) and vanishes at the other nodes. Then, the linear system (2.14) can be written as

$$\sum_{(x_k,y_l)\in\overline{D}\backslash\overline{\Omega}} h_{kl} \int_\Gamma \tilde{u}(e_{kl})\tilde{u}(e_{ij}) = \int_\Gamma \left[g - \tilde{u}(f)\right]\tilde{u}(e_{ij}), \quad \text{for } (x_i, y_j) \in \overline{D}\backslash\overline{\Omega}, \tag{2.15}$$

where $h_{kl} = h(x_k, y_l)$. Denoting the Fourier coefficients of e_{ij} by $\hat{e}_{ij,mn}$, $-M \leqslant m \leqslant M$, $-N \leqslant n \leqslant N$, we get from (2.9),

$$\hat{e}_{ij,mn} = \frac{\mathrm{e}^{-\mathrm{i}(mX_i+nY_j)}}{(2M+1)(2N+1)}.$$

Then, from (2.6), the Fourier coefficients $\hat{u}_{ij,mn}$ of $\tilde{u}(e_{ij})$ are given by

$$\hat{u}_{ij,mn} = \frac{-\mathrm{e}^{-\mathrm{i}(mX_i+nY_j)}}{(2M+1)(2N+1)} \frac{1}{((4\pi^2/a^2)m^2 + (4\pi^2/b^2)n^2 + \sigma)}, \tag{2.16}$$

and the value of $\tilde{u}(e_{ij})$ at the mesh node (x_k, y_l) can be approximated by

$$\tilde{u}(e_{ij})_{kl} = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{u}_{ij,mn}\mathrm{e}^{\mathrm{i}(mX_k+nY_l)}. \tag{2.17}$$

Since the mesh on D has been taken regular in the directions x and y, we can use a double fast Fourier transform for the calculation of the right-hand side in (2.7)–(2.10).

Below, we outline the steps of the proposed algorithm in which the embedding method which uses the Fourier approximations on a rectangle is associated with an optimal distributed control.

Numerical algorithm.

1. We extend f with zero at the mesh nodes in $\overline{D}\backslash\overline{\Omega}$ and calculate the Fourier coefficients $\hat{f}_{mn}$ given in (2.9).

2. Using (2.6) we get the Fourier coefficients $\hat{u}_{mn}$, and then using (2.8) we calculate $\tilde{u}(f)$ at the mesh nodes of $\overline{D}$.

3. Using (2.16), we calculate the coefficients $\hat{u}_{ij,mn}$ for the mesh nodes $(x_i, y_j) \in \overline{D}\backslash\overline{\Omega}$, and then using (2.17) we calculate the values of $\tilde{u}(e_{ij})$ at the mesh nodes of $\overline{D}$.

4. Using the computed values of $\tilde{u}(f)$ and $\tilde{u}(e_{ij})$ at the mesh nodes of $\overline{D}$, we calculate, by interpolation, the values of $\tilde{u}(f)$ and $\tilde{u}(e_{ij})$ at the mesh points of Γ, which are subsequently used in the numerical integration of the integrals appearing in the algebraic system (2.15). These boundary integrals have been calculated by the trapezoidal rule.

5. Using the solution of algebraic system (2.15), which gives the extension $\tilde{f}$ of f in $\overline{D}\backslash\overline{\Omega}$, and the values of f given in $\overline{\Omega}$, we get the values of $\tilde{u}$ at the mesh nodes of Ω from (2.9), (2.6) and (2.8).

Now, we make some remarks on this algorithm.

- The algebraic linear system (2.15) has a symmetric and full matrix. It can be solved either by an iterative method such as conjugate gradient method or by a direct method. We have applied below in the numerical tests the Gauss elimination method.
- Steps 3 and 4 of the algorithm are highly parallelizable on the multi-processor computing machines.
- In step 4, we use a linear interpolation in both directions x and y to obtain the values at the mesh points on Γ from the computed values at the mesh nodes of D. Assuming that a mesh point (x, y) on Γ lies in the cell defined by the four mesh nodes (x_1, y_1), (x_1, y_2), (x_2, y_1) and (x_2, y_2), we have linearly interpolated in the direction of x first the values corresponding to (x_1, y_1) and (x_2, y_1), and then the values corresponding to (x_1, y_2) and (x_2, y_2). Using these two computed values, we have made a linear interpolation in the direction of y.

In our numerical examples, the numerical integration in step 4 is carried out under the assumption that the boundary Γ is a polygon with the vertices at the mesh points and the two functions under the integral are linear between two consecutive mesh points.

3. Numerical results

In this section, we present numerical results for the solution of the Dirichlet problem (2.1) with $\sigma = 1.0$. As a fictitious domain we have taken the square $D = (0, a) \times (0, a)$ with $a = 1.386$. The domain Ω of the problem (2.1) is a hexagon with the center at $(a/2, a/2)$, whose sides are tangent to a circle of diameter 1.0. We have tested the algorithm given in the previous section for the problem which has the exact solution

$$u(x, y) = \frac{1}{4}\left[1 + \tanh\left(\alpha\left(x - \frac{a}{2}\right)\right)\right]\left[1 + \tanh\left(\alpha\left(y - \frac{a}{2}\right)\right)\right], \tag{3.1}$$

with $\alpha = 7.22$. The Dirichlet problem (2.1) with this as a solution has the inhomogeneous term f given by

$$\begin{aligned} f(x, y) = &-2\left[\frac{\alpha}{\cosh(\alpha(x - a/2))}\right]^2 \tanh\left(\alpha\left(x - \frac{a}{2}\right)\right)\left[1 + \tanh\left(\alpha\left(y - \frac{a}{2}\right)\right)\right] \\ &- 2\left[\frac{\alpha}{\cosh(\alpha(y - a/2))}\right]^2 \tanh\left(\alpha\left(y - \frac{a}{2}\right)\right)\left[1 + \tanh\left(\alpha\left(x - \frac{a}{2}\right)\right)\right] \\ &- \frac{\sigma}{4}\left[1 + \tanh\left(\alpha\left(x - \frac{a}{2}\right)\right)\right]\left[1 + \tanh\left(\alpha\left(y - \frac{a}{2}\right)\right)\right]. \end{aligned} \tag{3.2}$$

This numerical example is taken from [10] except that the problem is now translated from the square $(-a/2, a/2) \times (-a/2, a/2)$ to the square $(0, a) \times (0, a)$. The domains Ω and D are shown in figure 1. In [10], the authors have suggested that, in order to diminish the computing time, the extension of the nonhomogeneous term f should be sought not in the entire domain $\overline{D}\backslash\overline{\Omega}$, but only in a strip in $\overline{D}\backslash\overline{\Omega}$ surrounding Ω. The strip we have taken in our tests is bounded by the hexagonal boundaries of the domain Ω and a larger hexagon (plotted with dashed line in figure 1). This larger hexagon is concentric

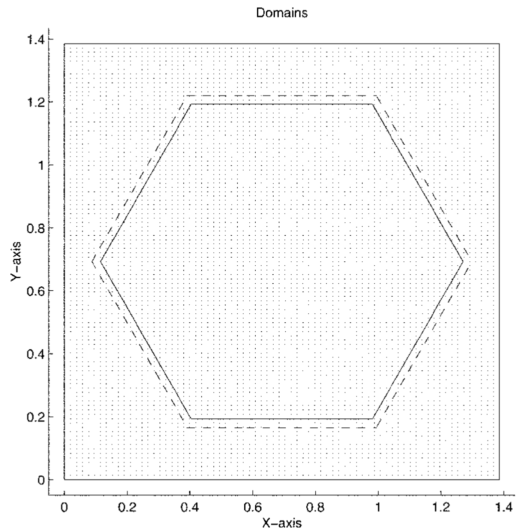

Figure 1. Hexagonal domain Ω is embedded within a rectangular domain D.

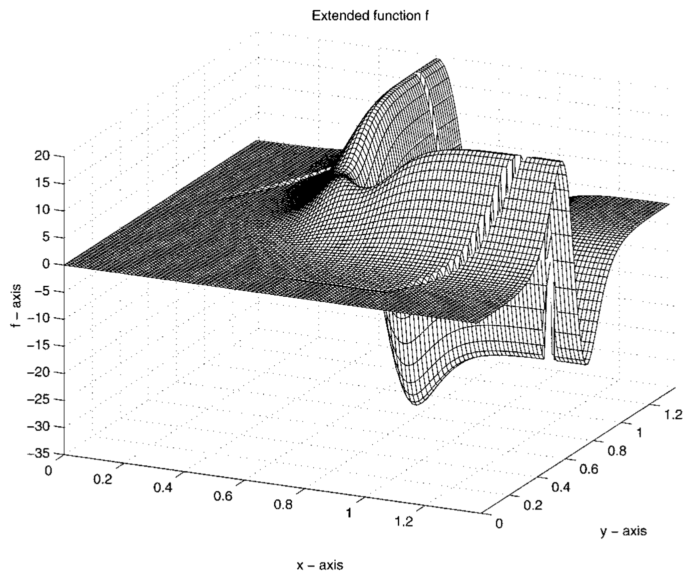

Figure 2. Extended function $\tilde{f}$ by the formula of f.

with Ω and circumscribed by a circle whose radius is 2δ larger than that of the circle circumscribing Ω, δ being the discretization step size in D in both directions, x and y. The values of $\tilde{f}$ outside of this strip are kept fixed. In order to assess the sensitivity of the solution on these values, two types of extensions of $\tilde{f}$ have been used in our numerical tests. In the first type, the values outside of the strip are taken to be zero. In the second type, the values outside the strip are computed with the same formula (3.2) which has been used for the domain Ω as well.

In figure 2 we have plotted $\tilde{f}$ with the extension given by (3.2) outside the strip and by zero inside the strip. In figure 3, the function u given by (3.1), which is the exact solution of our problem in Ω, is plotted over the whole domain D.

In the numerical tests, we have considered a mesh on Γ containing 720 equidistant points. At these mesh points we have calculated the absolute error between the boundary data g and the values of the computed solution $\tilde{u}$ for both choices of $\tilde{f}$ discussed above.

In tables 1 and 2 we show the maximum, minimum and average of the absolute errors over all the 720 mesh points, denoted $\mathrm{err}_{\mathrm{max}}$, $\mathrm{err}_{\mathrm{min}}$ and $\mathrm{err}_{\mathrm{ave}}$, respectively, for the two choices of the extension $\tilde{f}$. Also, we give in tables 3 and 4 the same errors computed at the mesh nodes of the domain Ω.

The number of nodes of the mesh on D in the directions of x and y has been denoted in the tables by n_x and n_y, respectively. The number of unknowns of the linear algebraic system (2.15), i.e. the number of mesh nodes contained in the strip surrounding Ω, is denoted by n_s in table 5.

We make the remark that the maximum error in the domain has been obtained in all the numerical tests at points near the boundary, and the minimum is reached for points in the central region of the domain. Comparison of the errors obtained in the two

Table 1
Errors on the boundary – $\tilde{f}$ extended outside of the strip by 0.

$n_x \times n_y$	$\text{err}_{\max}$	$\text{err}_{\min}$	err_{ave}
10 × 10	0.14778E+00	0.94081E−04	0.50276E−01
20 × 20	0.67147E−02	0.87660E−08	0.78654E−03
40 × 40	0.16700E−02	0.10361E−08	0.13981E−03
50 × 50	0.11432E−02	0.56247E−08	0.98537E−04
80 × 80	0.46858E−03	0.56326E−09	0.37898E−04
100 × 100	0.25132E−03	0.85283E−09	0.21895E−04

Table 2
Errors on the boundary – $\tilde{f}$ extended outside of the strip by the formula of f.

$n_x \times n_y$	$\text{err}_{\max}$	$\text{err}_{\min}$	err_{ave}
10 × 10	0.14778E+00	0.94045E−04	0.50276E−01
20 × 20	0.67147E−02	0.85004E−08	0.78673E−03
40 × 40	0.16696E−02	0.94994E−09	0.13976E−03
50 × 50	0.11432E−02	0.16270E−08	0.98697E−04
80 × 80	0.46858E−03	0.25962E-10	0.37900E−04
100 × 100	0.25132E−03	0.69470E−09	0.21916E−04

Table 3
Errors in the domain – $\tilde{f}$ extended outside of the strip by 0.

$n_x \times n_y$	$\text{err}_{\max}$	$\text{err}_{\min}$	err_{ave}
10 × 10	0.35503E+00	0.22904E−02	0.57183E−01
20 × 20	0.25551E−01	0.26373E−05	0.44161E−02
40 × 40	0.90539E−02	0.15511E−08	0.37815E−03
50 × 50	0.88295E−02	0.44237E−06	0.80264E−03
80 × 80	0.22470E−02	0.15956E−07	0.90335E−04
100 × 100	0.19046E−02	0.32612E−08	0.65072E−04

Table 4
Errors in the domain – $\tilde{f}$ extended outside of the strip by the formula of f.

$n_x \times n_y$	$\text{err}_{\max}$	$\text{err}_{\min}$	err_{ave}
10 × 10	0.35503E+00	0.22903E−02	0.57183E−01
20 × 20	0.25547E−01	0.18012E−05	0.44168E−02
40 × 40	0.90231E−02	0.14685E−06	0.38251E−03
50 × 50	0.89375E−02	0.16076E−06	0.82617E−03
80 × 80	0.22558E−02	0.52912E−07	0.90749E−04
100 × 100	0.19063E−02	0.55113E−08	0.64589E−04

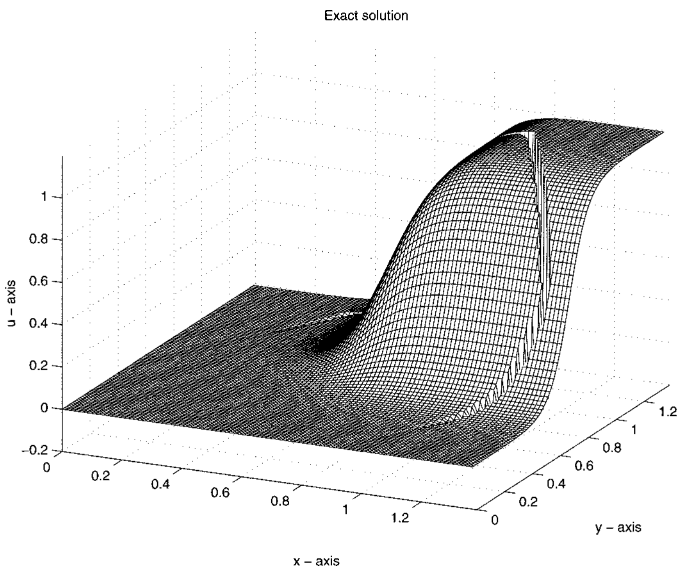

Figure 3. Exact solution u.

Table 5
Number of unknowns in the algebraic system (2.15).

$n_x \times n_y$	10×10	20×20	40×40	50×50	80×80	100×100
n_s	27	83	156	188	296	372

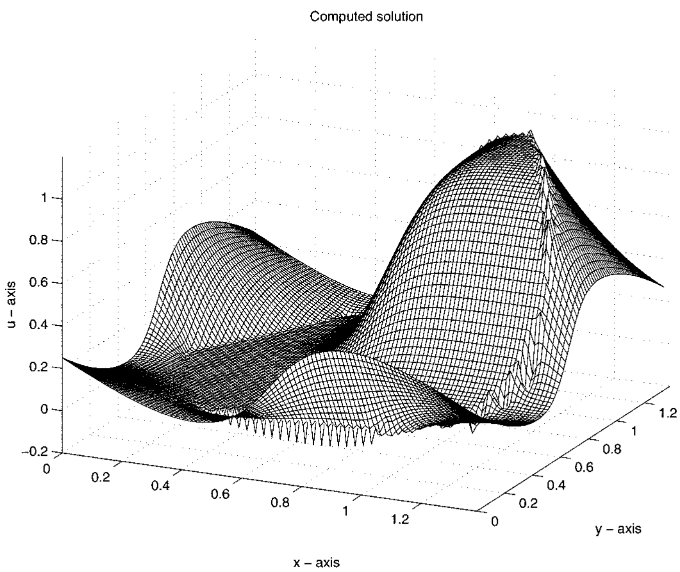

Figure 4. Computed solution $\tilde{u}$.

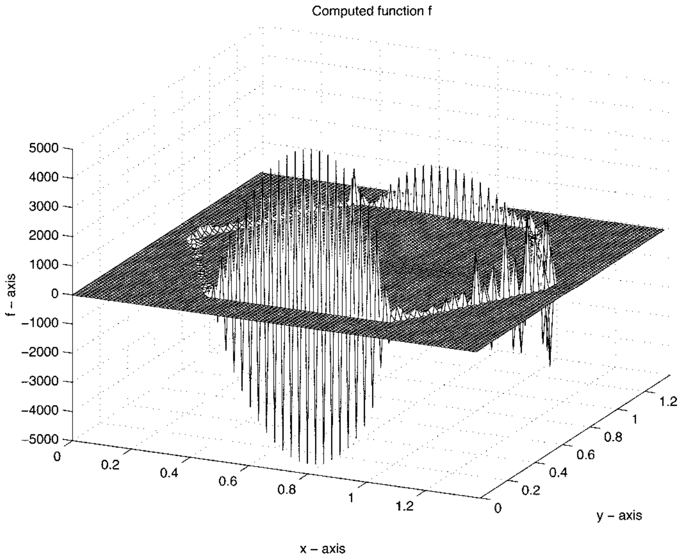

Figure 5. Computed function $\tilde{f}$.

types of extensions shows that they are almost equal. For both types of extensions we experimented with, there was very little difference in the results for the solution $\tilde{u}$. This possibly indicates that the sensitivity of the computed solution on the type of extension is likely to be negligible. For this reason, plots of $\tilde{u}$ and $\tilde{f}$ corresponding to the extension with zero of f only are shown in figures 4 and 5, respectively. Also, we notice that the error on the boundary Γ is less than that in the domain Ω in all of our numerical tests.

In figure 5, we have plotted the optimal nonhomogeneous term $\tilde{f}$ which we have obtained for problem (2.12) in the case $n_x = n_y = 100$. In figure 4, we have plotted the solution $\tilde{u}$ of the problem (2.2) corresponding to the optimal distributed control $\tilde{f}$. We see in figure 5 that the nonhomogeneous term $\tilde{f}$ has strong oscillations in the strip. Because of these large oscillations, the shape of $\tilde{f}$ in Ω and outside the strip appears to be flat in this figure. These oscillations have undesirable effects on the solution $\tilde{u}$ outside of the domain Ω as seen in figure 4. However, as we have seen in tables 1–4, the solution $\tilde{u}$ of the problem (2.2) approximates very well the boundary conditions g and the solution u of the problem (2.1).

4. Conclusions

In this paper we have proposed an embedding approach based on an optimal distributed control and Fourier spectral method for solving elliptic problems in arbitrary two-dimensional domains. Using this method one can solve elliptic problems in complex domains which may be either simply or multiply connected. The method can be easily applied to solve elliptic problems in three dimensions. In this method, an optimal extension of the nonhomogeneous term f is obtained by solving a linear algebraic system (see section 2). In the construction of this linear system, we use the fast Fourier transforms to find a periodic solution of the equation in the fictitious rectangular domain. The obtained algebraic linear system has a full matrix whose dimension can be made small by making use of a small strip around the domain Ω. Our numerical tests show that the numerical solution depends weakly on the extension of f outside of this strip. However, as we have seen in section 3, the numerical results show a good accuracy of the proposed method for relatively large mesh sizes.

In our opinion, there exists much room for the improvement of this method and we present in the following some topics of future research, which fall outside the scope of this paper. As mentioned at the beginning of section 2, problems of type (2.1) arise from the time discretization of the unsteady advection–diffusion problems which requires one to solve at each time step problems of type (2.1). In this case, the advection term is treated explicitly which appears in the source term f of equation (2.1). Consequently, the problem of the accurate computation of the derivatives of u should not be ignored when applying this proposed method to solving unsteady advection–diffusion problems. Even though, it appears from our numerical results that the extension of f outside the strip does not influence the results significantly, it is possible that perhaps some other extensions of f in step 1 of the numerical algorithm described in section 2 give better $\tilde{u}(f)$ at the step 2, and then the error near the boundary could be reduced.

Our numerical tests have shown that the error of the computed solution strongly depends on the dimension of the strip. The larger strips diminish the oscillations of the computed inhomogeneous term $\tilde{f}$, but because the embedding method leads in some sense to solving ill posed problems, the linear algebraic system becomes almost singular for large strips. Our numerical experiments have shown that the small strips give the best results. In order for this method to work with large strips as well, we will need to construct appropriate preconditioners for the algebraic systems resulting from this method.

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