Singular integral transforms and fast numerical algorithms

Prabir Daripa and Daoud Mashat

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

Received 6 November 1997; revised 28 May 1998
Communicated by M.H. Gutknecht

Fast algorithms for the accurate evaluation of some singular integral operators that arise in the context of solving certain partial differential equations within the unit circle in the complex plane are presented. These algorithms are generalizations and extensions of a fast algorithm of Daripa [11]. They are based on some recursive relations in Fourier space and the FFT (Fast Fourier Transform), and have theoretical computational complexity of the order $O(\log N)$ per point, where N^2 is the total number of grid points. An application of these algorithms to quasiconformal mappings of doubly connected domains onto annuli is presented in a follow-up paper.

Keywords: singular integrals, fast algorithm, Beltrami equation, complex variable

AMS subject classification: 65Y20, 65E05, 45P05, 44A20, 30E20, 17-08

1. Introduction

Many problems in applied mathematics require the evaluation of the singular integral transforms

$$T_m h(\sigma) = -\frac{1}{\pi} \iint_{B(0;1)} \frac{h(\zeta)}{(\zeta - \sigma)^m} \, d\xi \, d\eta, \quad \zeta = \xi + i\eta, \tag{1.1}$$

of a complex valued function h defined on $B(0;1) = \{z\colon\ |z| < 1\}$, for a suitable finite positive integer m so that the transform exists (see appendix B). For example, the general solution of the compressible fluid flow equations

$$u_{\bar{\sigma}} = \lambda u_\sigma \equiv h(\sigma) \tag{1.2}$$

in the hodograph plane is given by

$$u(\sigma) = T_1 h(\sigma) + g(\sigma) \tag{1.3}$$

in the subcritical regime (see Daripa [9]). In (1.2) λ is a Mach number dependent function (see [9]) and $g(\sigma)$ in (1.3) is a suitable analytic function determined by the boundary conditions. Unless otherwise specified, here and below $h(\sigma)$ depends on the solution u and its generalized derivatives, $\bar{\sigma}$, etc. This should be clear from the context even though such dependency will almost always be suppressed from the notation as in (1.2).

The function $h(\sigma)$ in (1.2) depends on the generalized derivative $u_\sigma(\sigma)$ given by

$$u_\sigma(\sigma) = T_2 h(\sigma) + g_\sigma(\sigma), \tag{1.4}$$

which follows from (1.3) (see [1,2]). Therefore, the numerical solution of equation (1.2) using the representation (1.3) requires evaluation of $T_m h(\sigma)$ for $m = 1, 2$. Evaluation of the operator $T_m h(\sigma)$ for $m > 2$ may be necessary if one requires to evaluate higher order derivatives of the function u, which is often the case in certain problems. In such cases ($m > 2$), existence of $T_m h(\sigma)$ can be guaranteed if $h(\sigma)$ is an analytic function (see appendix B), otherwise it is a hypersingular integral. The existence of such a hypersingular integral has to be understood in some appropriate finite part sense, which we do not address in this paper.

The equation (1.2) is known as the Beltrami equation and occurs in the context of other applied problems, most of which can be viewed as problems in quasiconformal mappings satisfying nonlinear partial differential equations of the following type [1,3–7,10,12,15,16]

$$u_{\bar\sigma} = h(\sigma, u, u_\sigma). \tag{1.5}$$

The Beltrami equation (1.2) is a special case of this equation. Other examples where integral operators (1.1) arise include problems in partial differential equations [5,6,8, 11,18–20], fluid mechanics [3,9] and electrostatics [17], to name just a few.

The integral equation approach of numerically solving equation (1.5) using the representation (1.3) requires computing the values of the integrals $T_1 h(\sigma)$ and $T_2 h(\sigma)$ at the discretization points. There are two main difficulties in the straightforward computation of these integrals using quadrature rules. Firstly, straightforward computation of each of these integrals requires an operation count of the order O(N^2) per point. This gives a net operation count of O(N^4) for N^2 grid points which is computationally very intensive for large N. Secondly, this method also gives poor accuracy due to the presence of the singularities in the integrand. The conventional desingularization technique [9] for evaluating these integrals is not effective in improving the accuracy because the function $h(\sigma)$ in (1.1) need not be an analytic function.

In Daripa [11], an efficient and accurate algorithm for rapid evaluation of the singular integral $T_1 h(\sigma)$ has been presented. Our goal in this paper is to generalize that algorithm providing all details of the theory to rapidly and accurately evaluate $T_m h(\sigma)$ for the following cases where the operator is known to exist (see appendix B).

(i) The complex valued function $h(\sigma)$ in (1.1) is Hölder continuous with exponent γ, $0 < \gamma < 1$. In this case, $T_1 h(\sigma)$ and $T_2 h(\sigma)$ are defined in the unit disk as a Cauchy principal value (see appendix B). The fast algorithms derived in this paper include the fast algorithm of Daripa presented for $T_1 h(\sigma)$ in [11] as a special case.

(ii) The complex valued function $h(\sigma)$ is analytic in the unit disk. In this case, $T_m h(\sigma)$ is defined in the unit disk for any finite positive integer m as a Cauchy principal value, and can be evaluated by the fast algorithm derived in this paper.

The method presented takes into account the convolution nature of these integrals and some of the properties of such convolution integrals in Fourier space. This process leads to a recursive algorithm in Fourier space that divides the entire domain into a collection of annular regions and expands the integral in Fourier series with radius dependent Fourier coefficients. A set of exact recursive relations are obtained which are then used to produce the Fourier coefficients of the integral. These recursive relations involve appropriate scaling of one-dimensional integrals in annular regions, which significantly improves the computational complexity. The desired integrals at all N^2 grid points are then easily obtained from the Fourier coefficients by the FFT (fast Fourier transform). The process of evaluation of these integrals has thus been optimized in this paper giving a net operation count of the order O(ln N) per point. For $N = 128$, this means *a reduction of over two thousand times*. It is worth emphasizing that the asymptotic computational complexity is independent of the number of integrals $T_m h(\sigma)$ whose values need to computed at each of the grid points. In fact, computing both integrals $T_1 h(\sigma)$ and $T_2 h(\sigma)$ costs little more than computing just $T_1 h(\sigma)$. Moreover, this algorithm has the added advantage of working in place, meaning that no additional memory storage is required beyond that of the initial data.

The effective use of representations (1.3) and (1.4) in solving (1.2) for generating quasiconformal mappings and for the design of subcritical airfoils has been demonstrated by Daripa (see [9,11,12,14]). The computational complexity of the various numerical algorithms used there can be further improved by the use of the fast algorithms presented here. Such efforts are currently in progress and will be reported in the future (see [14]). At this point, it is worth noting that this is an improved version of an unpublished paper of Daripa [13] which treats the case (i) mentioned above.

This paper is laid out as follows. In section 2 we present the mathematical foundation of fast algorithms for rapid evaluation of $T_m h(\sigma)$ within the unit circle. In section 3, the formal description of the fast algorithms is presented. In section 4, the algorithms are exemplified and validated. Finally, we summarize and conclude in section 5. In appendix A, we provide four propositions used in proving the lemmas of section 2 and in appendix B we show the existence of the integrals by estimating their upper bounds.

2. Evaluation of the T_m-operator

2.1. Theory for the evaluation of the T_m-operator

In this section, we present the theory needed to construct an efficient algorithm for the evaluation of the T_m-operator which is presented in section 3. Existence of these integrals is discussed in appendix B. For our purposes below, we let $r > \varepsilon > 0$,

$\delta > 0$ and $\sigma = r\mathrm{e}^{\mathrm{i}\alpha}$. We also introduce the following notations used in this section and in the appendices A and B.

$$B(\sigma;\delta) = \{z\colon\ |z-\sigma| < \delta\}, \qquad \overline{B}(\sigma;\delta) = \{z\colon\ |z-\sigma| \leqslant \delta\},$$
$${}_0\Omega_r = B(0;r), \qquad {}_{r-\varepsilon}\Omega_{r+\varepsilon} = B(0;r+\varepsilon) - B(0;r-\varepsilon),$$
$${}_{r-\varepsilon}\Omega^*_{r+\varepsilon} = {}_{r-\varepsilon}\Omega_{r+\varepsilon} - B(\sigma;\varepsilon),$$

and

$${}_r\Omega_1 = B(0;1) - B(0;r).$$

Below we present the main theorem and its proof. The theorem treats the cases (i) and (ii) mentioned in section 1.

Theorem 2.1. If $T_m h(\sigma)$ exists in the unit disk as a Cauchy principal value, and $h(r\mathrm{e}^{\mathrm{i}\alpha}) = \sum_{n=-\infty}^{\infty} h_n(r)\mathrm{e}^{\mathrm{i}n\alpha}$, then the nth Fourier series coefficient $S_{n,m}(r)$ of $T_m h(r\mathrm{e}^{\mathrm{i}\alpha})$ can be written as

$$S_{n,m}(r) = \begin{cases} C_{n,m}(r) + B_{n,m}(r), & r \neq 0, \\ 0, & r = 0 \text{ and } n \neq 0, \\ S_{0,m}(0), & r = 0 \text{ and } n = 0, \end{cases} \tag{2.1}$$

where

$$C_{n,m}(r) = \begin{cases} \dfrac{2(-1)^{m+1}}{r^{m-1}} \dbinom{-n-1}{m-1} \displaystyle\int_0^r \left(\frac{r}{\rho}\right)^{m+n-1} h_{m+n}(\rho)\,\mathrm{d}\rho, & n \leqslant -m, \\ 0, & -m < n < 0, \\ -\dfrac{2}{r^{m-1}} \dbinom{m+n-1}{m-1} \displaystyle\int_r^1 \left(\frac{r}{\rho}\right)^{m+n-1} h_{m+n}(\rho)\,\mathrm{d}\rho, & n \geqslant 0, \end{cases} \tag{2.2}$$

and $B_{n,m}(r)$ and $S_{0,m}(0)$ are defined as follows.

Case 1. If $h(\sigma)$ is Hölder continuous in the unit disk with exponent γ, $0 < \gamma < 1$, and $m = 1$ or 2, then

$$S_{0,m}(0) = -2\lim_{\varepsilon\to 0}\int_\varepsilon^1 \rho^{1-m} h_m(\rho)\,\mathrm{d}\rho, \tag{2.3}$$

$$B_{n,m}(r) = \begin{cases} 0, & m = 1, \\ h_{n+2}(r), & m = 2. \end{cases} \tag{2.4}$$

Case 2. If $h(\sigma)$ is analytic in the unit disk and m is a finite positive integer, then

$$S_{0,m}(0) = -h_m(r=1), \tag{2.5}$$

$$B_{n,1}(r) = 0, \tag{2.6}$$

and for $m \geqslant 2$

$$B_{n,m}(r) = \begin{cases} 0, & n < -1,\ n \neq -m, \\ (-1)^m r^{2-m} h_0(r), & n = -m, \\ \binom{m+n-1}{m-2} r^{2-m} h_{m+n}(r), & n \geqslant -1. \end{cases} \tag{2.7}$$

The proof of this theorem is based on evaluating the integral $T_m h(\sigma)$ in a domain $B(0,1) - B(\sigma;\varepsilon)$ with $\varepsilon \to 0$. Noting that $B(0,1) - B(\sigma;\varepsilon) = {}_0\mathbf{\Omega}_{r-\varepsilon} \cup {}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon} \cup {}_{r+\varepsilon}\mathbf{\Omega}_1$, the integral in domain $B(0,1) - B(\sigma,\varepsilon)$ is obtained by integration in these three domains. The lemmas below deal with the integration over the domain ${}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}$ except for lemmas 2.5 and 2.7 which deal with the situation when $\sigma = 0$ in (1.1).

The lemmas 2.1–2.5 below are used for case 1 and lemmas 2.6 and 2.7 are used for case 2. In particular, lemmas 2.1–2.4 are used for evaluating the integral

$$\lim_{\varepsilon \to 0} \iint_{{}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}} \frac{h(\sigma)}{\zeta - \sigma}\, \mathrm{d}\xi\, \mathrm{d}\eta$$

for case 1 and lemma 2.6 is used for evaluating the same integral for case 2. Lemma 2.5 is used for obtaining $S_{n,m}(0)$ for case 1 (see equations (2.1) and (2.3)) and lemma 2.7 is used for obtaining $S_{0,m}(0)$ for case 2 (equation (2.5)), respectively.

Lemma 2.1. If $h(\sigma)$ is Hölder continuous with exponent γ, $0 < \gamma < 1$, and constant K, then

$$\lim_{\varepsilon \to 0} \iint_{{}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}} \frac{h(\zeta)}{\zeta - \sigma}\, \mathrm{d}\xi\, \mathrm{d}\eta = 0. \tag{2.8}$$

Proof. It follows from proposition 1 in appendix A that

$$\lim_{\varepsilon \to 0} \iint_{{}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}} \frac{h(\sigma)}{\zeta - \sigma}\, \mathrm{d}\xi\, \mathrm{d}\eta = 0. \tag{2.9}$$

Therefore,

$$\begin{aligned} &\left| \lim_{\varepsilon \to 0} \iint_{{}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}} \frac{h(\zeta)}{\zeta - \sigma}\, \mathrm{d}\xi\, \mathrm{d}\eta \right| \\ &\quad = \left| \lim_{\varepsilon \to 0} \iint_{{}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}} \frac{h(\zeta) - h(\sigma)}{\zeta - \sigma}\, \mathrm{d}\xi\, \mathrm{d}\eta \right| \leqslant \lim_{\varepsilon \to 0} \iint_{{}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}} \frac{|h(\zeta) - h(\sigma)|}{|\zeta - \sigma|}\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &\quad \leqslant K \lim_{\varepsilon \to 0} \iint_{{}_{r-\varepsilon}\mathbf{\Omega}^*_{r+\varepsilon}} |\zeta - \sigma|^{\gamma - 1}\, \mathrm{d}\xi\, \mathrm{d}\eta \leqslant K \lim_{\varepsilon \to 0} \varepsilon^{\gamma-1} \left\{ \pi(r+\varepsilon)^2 - \pi(r-\varepsilon)^2 - \pi\varepsilon^2 \right\} \\ &\quad = 0. \end{aligned} \tag{2.10}$$

Lemma 2.1 follows from (2.10). □

Lemma 2.2. If $h(\sigma)$ is Hölder continuous with exponent γ, $0 < \gamma < 1$, and constant K, then

$$\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta = \lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\sigma)}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{2.11}$$

Proof. We have

$$\begin{aligned}
&\left|\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta - \lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\sigma)}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\right| \\
&\quad\leqslant \lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{|h(\zeta)-h(\sigma)|}{|\zeta-\sigma|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad\leqslant K\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0.
\end{aligned} \tag{2.12}$$

We have used proposition 2 from appendix A in arriving at the last inequality above. Lemma 2.2 follows from (2.12). □

Lemma 2.3. If $h(r\,\mathrm{e}^{\mathrm{i}\alpha}) = \sum_{n=-\infty}^{\infty} h_n(r)\,\mathrm{e}^{\mathrm{i}n\alpha}$, then

$$-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\sigma)}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta = \sum_{n=-\infty}^{\infty} h_{n+2}(r)\,\mathrm{e}^{\mathrm{i}n\alpha}, \quad \sigma \neq 0. \tag{2.13}$$

Proof. Using proposition 1 from appendix A, we have

$$-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\sigma)}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{h(\sigma)r^2}{\sigma^2} = \sum_{n=-\infty}^{\infty} h_{n+2}(r)\,\mathrm{e}^{\mathrm{i}n\alpha}. \tag{2.14}$$

□

Lemma 2.4. If $h(\sigma)$ is Hölder continuous with exponent γ, $0 < \gamma < 1$, and constant K, and

$$h\big(r\,\mathrm{e}^{\mathrm{i}\alpha}\big) = \sum_{n=-\infty}^{\infty} h_n(r)\,\mathrm{e}^{\mathrm{i}n\alpha},$$

then

$$-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta = \sum_{n=-\infty}^{\infty} h_{n+2}(r)\,\mathrm{e}^{\mathrm{i}n\alpha}, \quad \sigma \neq 0. \tag{2.15}$$

Proof. The lemma follows from lemmas 2.2 and 2.3. □

Lemma 2.5. If $h(r\,\mathrm{e}^{\mathrm{i}\alpha}) = \sum_{n=-\infty}^{\infty} h_n(r)\,\mathrm{e}^{\mathrm{i}n\alpha}$, then

$$S_{n,m}(0) = 0, \quad n \neq 0, \quad \text{and} \quad S_{0,m}(0) = -2\lim_{\varepsilon\to 0}\int_\varepsilon^1 \rho^{1-m} h_m(\rho)\,\mathrm{d}\rho. \tag{2.16}$$

Proof. Since $T_m h(0)$ is a constant (see equation (1.1)), it follows that all of its Fourier coefficients are zero except for $S_{0,m}(0)$, which is equal to $T_m h(0)$ and can be evaluated as follows:

$$\begin{aligned} S_{0,m}(0) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{B(0;1)-B(0;\varepsilon)} \frac{h(\zeta)}{\zeta^m}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{B(0;1)-B(0;\varepsilon)} \frac{h(\rho\,\mathrm{e}^{\mathrm{i}\theta})}{\rho^m \mathrm{e}^{\mathrm{i}m\theta}}\rho\,\mathrm{d}\theta\,\mathrm{d}\rho \\ &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_\varepsilon^1 \rho^{1-m}\sum_{n=-\infty}^{\infty} h_n(\rho)\left(\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta\right)\mathrm{d}\rho \\ &= -2\lim_{\varepsilon\to 0}\int_\varepsilon^1 \rho^{1-m} h_m(\rho)\,\mathrm{d}\rho. \end{aligned} \tag{2.17}$$

□

Lemma 2.6. If $h(\sigma)$ is an analytic function in the neighborhood of a unit disk, then for $m \geqslant 2$ and $|\sigma| \neq 0$

$$-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta = \sum_{n=-\infty}^{\infty} B_{n,m}(r)\,\mathrm{e}^{\mathrm{i}n\alpha}, \tag{2.18}$$

where

$$B_{n,m}(r) = \begin{cases} 0, & n < -1, n \neq -m, \\ (-1)^m r^{2-m} h_0(r), & n = -m, \\ \binom{m+n-1}{m-2} r^{2-m} h_{m+n}(r), & n \geqslant -1. \end{cases} \tag{2.19}$$

Proof. Using Green's theorem, we get

$$\begin{aligned} &\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad = \frac{1}{2\mathrm{i}}\int_{{}_{r-\varepsilon}\partial\Omega^*{}_{r+\varepsilon}} \frac{h(\zeta)\bar\zeta}{(\zeta-\sigma)^m}\,\mathrm{d}\zeta \\ &\quad = \frac{1}{2\mathrm{i}}\left\{\int_{\partial B(0;r+\varepsilon)} \frac{h(\zeta)\bar\zeta}{(\zeta-\sigma)^m}\,\mathrm{d}\zeta - \int_{\partial B(0;r-\varepsilon)} \frac{h(\zeta)\bar\zeta}{(\zeta-\sigma)^m}\,\mathrm{d}\zeta\right. \end{aligned}$$

$$
\begin{aligned}
&- \int_{\partial B(\sigma;\varepsilon)} \frac{h(\zeta)\bar{\zeta}}{(\zeta-\sigma)^m}\,\mathrm{d}\zeta \Bigg\} \\
&= \frac{1}{2\mathrm{i}}\Bigg\{(r+\varepsilon)^2 \int_{\partial B(0;r+\varepsilon)} \frac{h(\zeta)}{\zeta(\zeta-\sigma)^m}\,\mathrm{d}\zeta - (r-\varepsilon)^2 \int_{\partial B(0;r-\varepsilon)} \frac{h(\zeta)}{\zeta(\zeta-\sigma)^m}\,\mathrm{d}\zeta \\
&- \int_0^{2\pi} \frac{h(\sigma+\varepsilon\,\mathrm{e}^{\mathrm{i}\theta})(\bar{\sigma}+\varepsilon\,\mathrm{e}^{-\mathrm{i}\theta})}{\varepsilon^m \mathrm{e}^{\mathrm{i}m\theta}}\,\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\Bigg\} \\
&= \frac{1}{2\mathrm{i}}\Bigg\{2\pi\mathrm{i}(r+\varepsilon)^2\left[\frac{h(0)}{(-\sigma)^m} + \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\zeta^{m-1}}\left(\frac{h(\zeta)}{\zeta}\right)\bigg|_{\zeta=\sigma}\right] \\
&- 2\pi\mathrm{i}(r-\varepsilon)^2\frac{h(0)}{(-\sigma)^m} - \frac{\mathrm{i}}{\varepsilon^{m-1}}\int_0^{2\pi}\sum_{s=0}^{\infty}\frac{h_s(r)}{r^s}\left(\sigma+\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\right)^s \\
&\times\left(\bar{\sigma}\,\mathrm{e}^{-(m-1)\mathrm{i}\theta} + \varepsilon\,\mathrm{e}^{-\mathrm{i}m\theta}\right)\mathrm{d}\theta\Bigg\} \\
&= \frac{1}{2\mathrm{i}}\Bigg\{2\pi\mathrm{i}(r+\varepsilon)^2\left[\frac{h(0)}{(-\sigma)^m} + \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\zeta^{m-1}}\left(\frac{h(\zeta)}{\zeta}\right)\bigg|_{\zeta=\sigma}\right] \\
&- 2\pi\mathrm{i}(r-\varepsilon)^2\frac{h(0)}{(-\sigma)^m} - \frac{\mathrm{i}}{\varepsilon^{m-1}}\sum_{s=m-1}^{\infty}\frac{h_s(r)}{r^s}\left[2\pi\bar{\sigma}\binom{s}{m-1}\sigma^{s-m+1}\varepsilon^{m-1}\right] \\
&- \frac{\mathrm{i}}{\varepsilon^{m-1}}\sum_{s=m}^{\infty}\frac{h_s(r)}{r^s}\left[2\pi\binom{s}{m}\sigma^{s-m}\varepsilon^{m+1}\right]\Bigg\}.
\end{aligned}
\tag{2.20}
$$

Therefore,

$$
\begin{aligned}
&-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad= -\frac{1}{2\pi\mathrm{i}}\Bigg\{2\pi\mathrm{i}r^2\left[\frac{h(0)}{(-\sigma)^m} + \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\zeta^{m-1}}\left(\frac{h(\zeta)}{\zeta}\right)\bigg|_{\zeta=\sigma}\right] \\
&\quad- 2\pi\mathrm{i}r^2\frac{h(0)}{(-\sigma)^m} - 2\pi\mathrm{i}\bar{\sigma}\sum_{s=m-1}^{\infty}\frac{h_s(r)}{r^s}\binom{s}{m-1}\sigma^{s-m+1}\Bigg\} \\
&\quad= -\Bigg\{\frac{r^2}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\zeta^{m-1}}\left(\frac{h(\zeta)}{\zeta}\right)\bigg|_{\zeta=\sigma} \\
&\quad- \bar{\sigma}\sum_{s=m-1}^{\infty}\frac{h_s(r)}{r^s}\binom{s}{m-1}\sigma^{s-m+1}\Bigg\}.
\end{aligned}
\tag{2.21}
$$

Using proposition 3 from appendix A in (2.21), the Fourier coefficients $B_{n,m}(r)$ of the integral operator on the left-hand side of (2.21), as defined in (2.18), are then given by

$$
\begin{aligned}
B_{n,m}(r) &= -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\alpha}\Biggl\{\frac{r^2}{(m-1)!}\sum_{s=0}^{\infty}\frac{h_s(r)}{r^m}(s-1)(s-2)\cdots \\
&\quad \times \bigl(s-(m-1)\bigr)\,\mathrm{e}^{\mathrm{i}(s-m)\alpha} - \sum_{s=m-1}^{\infty}\frac{h_s(r)}{r^{m-2}}\binom{s}{m-1}\mathrm{e}^{\mathrm{i}(s-m)\alpha}\Biggr\}\,\mathrm{d}\alpha \\
&= -\frac{1}{2\pi}\Biggl\{\frac{r^2}{(m-1)!}\sum_{s=0}^{\infty}\frac{h_s(r)}{r^m}(s-1)(s-2)\cdots \\
&\quad \times \bigl(s-(m-1)\bigr)\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}n\alpha}\,\mathrm{e}^{\mathrm{i}(s-m)\alpha}\,\mathrm{d}\alpha \\
&\quad - \sum_{s=m-1}^{\infty}\frac{h_s(r)}{r^{m-2}}\binom{s}{m-1}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}n\alpha}\,\mathrm{e}^{\mathrm{i}(s-m)\alpha}\,\mathrm{d}\alpha\Biggr\} \\
&= \begin{cases}
-\dfrac{1}{2\pi}\Biggl[\dfrac{2\pi r^2}{(m-1)!}\dfrac{h_{m+n}(r)}{r^m}(m+n-1)(m+n-2)\cdots & \\
\quad\times\bigl((m+n)-(m-1)\bigr) - \dfrac{2\pi h_{m+n}(r)}{r^{m-2}}\dbinom{m+n}{m-1}\Biggr], & n \geqslant -1, \\
0, & n < -1 \text{ and } \\
 & n \neq -m, \\
(-1)^m r^{2-m} h_0(r), & n = -m,
\end{cases} \\
&= \begin{cases}
\dbinom{m+n-1}{m-2} r^{2-m} h_{n+m}(r), & n \geqslant -1, \\
0, & n < -1 \text{ and } n \neq m, \\
(-1)^m r^{2-m} h_0(r), & n = -m.
\end{cases}
\end{aligned}
\tag{2.22}
$$

$\square$

Lemma 2.7. If $h(\sigma)$ is an analytic function in the neighborhood of the unit disk, then the Fourier coefficient $S_{0,m}(0) = T_m h(0)$ is given by

$$S_{0,m}(0) = -h_m(1). \tag{2.23}$$

Proof. Using Green's theorem, the Cauchy principal value of $T_m h(\sigma = 0)$ (see equation (1.1)) can be evaluated as follows:

$$
\begin{aligned}
T_m h(0) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{B(0;1)-B(0;\varepsilon)}\frac{h(\zeta)}{\zeta^m}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= -\frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0}\Biggl\{\int_{\partial B(0;1)}\frac{h(\zeta)\bar{\zeta}}{\zeta^m}\,\mathrm{d}\zeta - \int_{\partial B(0;\varepsilon)}\frac{h(\zeta)\bar{\zeta}}{\zeta^m}\,\mathrm{d}\zeta\Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon\to 0}\left\{\int_{\partial B(0;1)} \frac{h(\zeta)}{\zeta^{m+1}}\,\mathrm{d}\zeta - \int_0^{2\pi} \frac{h(\varepsilon\,\mathrm{e}^{\mathrm{i}\theta})\varepsilon\,\mathrm{e}^{-\mathrm{i}\theta}\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}}{\varepsilon^m\,\mathrm{e}^{\mathrm{i}m\theta}}\,\mathrm{d}\theta\right\}\\
&= -\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon\to 0}\left\{\frac{2\pi\mathrm{i}}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\zeta^m}h(\zeta)\bigg|_{\zeta=0} - \frac{\mathrm{i}}{\varepsilon^{m-2}}\int_0^{2\pi} h\big(\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\big)\,\mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta\right\}\\
&= -\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon\to 0}\left\{2\pi\mathrm{i}h_m(1) - \frac{\mathrm{i}}{\varepsilon^{m-2}}\sum_{n=0}^{\infty}\int_0^{2\pi} h_n(1)\varepsilon^n\mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta\right\}\\
&= -\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon\to 0}\left\{2\pi\mathrm{i}h_m(1) - \frac{2\pi\mathrm{i}}{\varepsilon^{m-2}}h_m(1)\varepsilon^m\right\}\\
&= -h_m(1). \qquad (2.24)
\end{aligned}
$$

□

Now we have all the lemmas necessary to prove theorem 2.1 which follows.

Proof of theorem 2.1. We rewrite $T_m h(\sigma)$ as a Cauchy principal value and divide the domain of integration into three subdomains for the purpose of integration:

$$
\begin{aligned}
T_m h(\sigma) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{B(0;1)-B(\sigma;\varepsilon)} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\left\{\iint_{{}_0\Omega_{r-\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta + \iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta\right.\\
&\qquad \left. + \iint_{{}_{r+\varepsilon}\Omega_1} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta\right\}. \qquad (2.25)
\end{aligned}
$$

The Fourier series coefficient $S_{n,m}(r)$ of $T_m h(r\,\mathrm{e}^{\mathrm{i}\alpha})$ is given by

$$
\begin{aligned}
S_{n,m}(r) &= -\frac{1}{2\pi^2}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\alpha}\lim_{\varepsilon\to 0}\left\{\iint_{{}_0\Omega_{r-\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta\right.\\
&\qquad \left. + \iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta + \iint_{{}_{r+\varepsilon}\Omega_1} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta\right\}\mathrm{d}\alpha\\
&= -\frac{1}{2\pi^2}\left\{\iint_{{}_0\Omega_r} h(\zeta)\int_0^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}n\alpha}}{(\zeta-\sigma)^m}\,\mathrm{d}\alpha\,\mathrm{d}\xi\,\mathrm{d}\eta\right.\\
&\qquad \left. + \iint_{{}_r\Omega_1} h(\zeta)\int_0^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}n\alpha}}{(\zeta-\sigma)^m}\,\mathrm{d}\alpha\,\mathrm{d}\xi\,\mathrm{d}\eta\right\}\\
&\qquad + \frac{1}{2\pi}\int_0^{2\pi}\left\{-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta\right\}\mathrm{e}^{-\mathrm{i}n\alpha}\,\mathrm{d}\alpha
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2\pi^2}\left\{\iint_{{}_0\Omega_r} h(\zeta)Q_{n,m}(r,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right.\\
&\quad \left. + \iint_{{}_r\Omega_1} h(\zeta)Q_{n,m}(r,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right\} + B_{n,m}(r),
\end{aligned} \tag{2.26}
$$

where

$$
Q_{n,m}(r,\zeta) = \int_0^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}n\alpha}}{(\zeta-\sigma)^m}\,\mathrm{d}\alpha, \tag{2.27}
$$

and $B_{n,m}(r)$ is the nth Fourier coefficient of

$$
-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta
$$

as defined earlier in the text. From proposition 4 in appendix A we have

$$
Q_{n,m}(r,\zeta) = \begin{cases}
2(-1)^m\pi r^n\binom{-n-1}{m-1}\zeta^{-(m+n)}, & n \leqslant -m, |\zeta| < r,\\
0, & n > -m, |\zeta| < r,\\
0, & n < 0, |\zeta| > r,\\
2\pi r^n\binom{m+n-1}{m-1}\zeta^{-(m+n)}, & n \geqslant 0, |\zeta| > r.
\end{cases} \tag{2.28}
$$

It is convenient to rewrite (2.26) as

$$
S_{n,m}(r) = C_{n,m}(r) + B_{n,m}(r) \tag{2.29}
$$

with $C_{n,m}(r)$ given by

$$
\begin{aligned}
C_{n,m}(r) &= -\frac{1}{2\pi^2}\left\{\iint_{{}_0\Omega_r} h(\zeta)Q_{n,m}(r,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \iint_{{}_r\Omega_1} h(\zeta)Q_{n,m}(r,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right\}\\
&= \begin{cases}
\dfrac{(-1)^{m+1}}{\pi} r^n\dbinom{-n-1}{m-1}\displaystyle\iint_{{}_0\Omega_r} h(\zeta)\zeta^{-(m+n)}\,\mathrm{d}\xi\,\mathrm{d}\eta, & n \leqslant m,\\
0, & -m < n < 0,\\
-\dfrac{r^n}{\pi}\dbinom{m+n-1}{m-1}\displaystyle\iint_{{}_r\Omega_1} h(\zeta)\zeta^{-(m+n)}\,\mathrm{d}\xi\,\mathrm{d}\eta, & n \geqslant 0,
\end{cases}\\
&= \begin{cases}
\dfrac{2(-1)^{m+1}}{r^{m-1}}\dbinom{-n-1}{m-1}\displaystyle\int_0^r \left(\frac{r}{\rho}\right)^{m+n-1} h_{m+n}(\rho)\,\mathrm{d}\rho, & n \leqslant -m,\\
0, & -m < n < 0,\\
-\dfrac{2}{r^{m-1}}\dbinom{m+n-1}{m-1}\displaystyle\int_r^1 \left(\frac{r}{\rho}\right)^{m+n-1} h_{m+n}(\rho)\,\mathrm{d}\rho, & n \geqslant 0.
\end{cases}
\end{aligned} \tag{2.30}
$$

Theorem (2.1) follows from (2.29), (2.30) and lemmas (2.1)–(2.7).

Use of theorem 2.1 in constructing a fast algorithm to evaluate the integral $T_m h(\sigma)$ requires the following recursive relations which follow from the theorem itself.

2.2. Recursive relations

Corollary 2.1. It follows from (2.2) that $C_{n,m}(1) = 0$ for $n \geqslant 0$, and $C_{n,m}(0) = 0$ for $n \leqslant -m$. We repeat from (2.2) that $C_{n,m}(r) = 0$ for $-m < n < 0$ for all values of r in the domain.

Corollary 2.2. If $r_j > r_i$ and

$$C_{n,m}^{i,j} = \begin{cases} \dfrac{2(-1)^{m+1}}{r_j^{m-1}} \dbinom{-n-1}{m-1} \displaystyle\int_{r_i}^{r_j} \left(\frac{r_j}{\rho}\right)^{m+n-1} h_{m+n}(\rho)\,\mathrm{d}\rho, & n \leqslant -m, \\ \dfrac{2}{r_i^{m-1}} \dbinom{m+n-1}{m-1} \displaystyle\int_{r_i}^{r_j} \left(\frac{r_i}{\rho}\right)^{m+n-1} h_{m+n}(\rho)\,\mathrm{d}\rho, & n \geqslant 0, \end{cases} \tag{2.31}$$

then

$$C_{n,m}(r_j) = \left(\frac{r_j}{r_i}\right)^n C_{n,m}(r_i) + C_{n,m}^{i,j}, \quad n \leqslant -m, \tag{2.32}$$

$$C_{n,m}(r_i) = \left(\frac{r_i}{r_j}\right)^n C_{n,m}(r_j) - C_{n,m}^{i,j}, \quad n \geqslant 0. \tag{2.33}$$

Proof. The proof is straightforward. The corollary 2.2 follows directly from simple manipulation of (2.2) and using (2.31). □

Corollary 2.3. Let $0 = r_1 < r_2 < \cdots < r_M = 1$, then

$$C_{n,m}(r_l) = \begin{cases} \sum_{i=2}^{l} \left(\dfrac{r_l}{r_i}\right)^n C_{n,m}^{i-1,i} & \text{for } n \leqslant -m \text{ and } l = 2, \ldots, M, \\ -\sum_{i=l}^{M-1} \left(\dfrac{r_l}{r_i}\right)^n C_{n,m}^{i,i+1} & \text{for } n \geqslant 0 \text{ and } l = 1, \ldots, M-1. \end{cases} \tag{2.34}$$

Proof. The proof is straightforward. Repeated application of (2.32) and $C_{n,m}(0) = 0$ for $n \leqslant -m$ (see corollary 2.1) gives the first part of the corollary. Similarly, repeated application of (2.33) and $C_{n,m}(1) = 0$ for $n \geqslant 0$ (see corollary 2.1) gives the second part of the corollary. □

3. Formal description of the fast algorithm

Recall that the unit disk $\overline{B}(0;1)$ is discretized using $N \times M$ lattice points with N equidistant points in the circular direction and M equidistant points in the radial direction. The following is a formal description of the fast algorithm useful for programming purposes.

Algorithm 3.1.
Input: $m \geqslant 1$, M, N and $h(r_l \mathrm{e}^{2\pi \mathrm{i} k/N})$, $l \in [1, M]$, $k \in [1, N]$.
Output: $T_m h(r_l \mathrm{e}^{2\pi \mathrm{i} k/N})$, $l \in [1, M]$, $k \in [1, N]$.
Step 1. Set $K = N/8$, $r_1 = 0$, and $r_M = 1$.
Step 2. Compute the Fourier coefficients $h_n(r_l)$, $\forall\, l \in [1, M]$ and $n \in [-K + m, K]$, from known values of $h(r_l \mathrm{e}^{2\pi \mathrm{i} k/N})$, $k = 1, 2, \ldots, N$, using the FFT.
Step 3. Compute $C_{n,m}^{i,i+1}$, $\forall i \in [1, M-1]$ and $n \in \{[-K, -m] \cup [0, K - m]\}$, using equation (2.31).
Step 4.
Note: Compute $C_{n,m}(r_l)$, $\forall l \in [1, M]$ and $n \in [-K, K-m]$, using corollaries 2.1–2.3.
  set $C_{n,m}(r_1) = 0\ \forall n \in [-K, -m]$
  do $n = -K, \ldots, -m$
    do $l = 2, \ldots, M$

$$C_{n,m}(r_l) = \left(\frac{r_l}{r_{l-1}}\right)^n C_{n,m}(r_{l-1}) + C_{n,m}^{l-1,l}$$

    enddo
  enddo
  set $C_{n,m}(r_M) = 0\ \forall n \in [0, K - m]$
  do $n = 0, 1, \ldots, K - m$
    do $l = M - 1, \ldots, 1$

$$C_{n,m}(r_l) = \left(\frac{r_l}{r_{l+1}}\right)^n C_{n,m}(r_{l+1}) - C_{n,m}^{l,l+1}$$

    enddo
  enddo
  If $m > 1$, **then**
  do $n = -m + 1, \ldots, -1$
    do $l = 1, \ldots, M$

$$C_{n,m}(r_l) = 0$$

    enddo
  enddo
  end if
Step 5.
Note: Compute $B_{n,m}(r_l)$, $\forall l \in [2, M]$ and $n \in [-K, K - m]$, using equations (2.4), (2.6) and (2.7).
  If $m = 1$, **then**
    set $B_{n,m}(r_l) = 0\ \forall l \in [2, M]$ and $n \in [-K, K - m]$
  else

Compute $B_{n,m}(r_l)$, $\forall l \in [2, M]$ and $n \in [-K, K-m]$, using (2.4) for case 1 and (2.6) for case 2 (see theorem 2.1).

end if

Step 6.

Note: Compute the Fourier coefficients $S_{n,m}(r_l)$, $\forall l \in [2, M]$ and $n \in [-K, K-m]$, using equation (2.1).

do $n = -K, \dots, K-m$

do $l = 2, \dots, M$

$$S_{n,m}(r_l) = B_{n,m}(r_l) + C_{n,m}(r_l)$$

enddo

enddo

Compute $S_{n,m}(0)$, $n \in [-K, K-m]$ from (2.1) using (2.3) for case 1 and (2.5) for case 2.

Step 7. Compute

$$T_m h\big(r_l \mathrm{e}^{2\pi \mathrm{i} k/N}\big) = \sum_{n=-K}^{K-m} S_{n,m}(r_l)\, \mathrm{e}^{2\pi \mathrm{i} kn/N}, \quad \forall l \in [1, M] \text{ and } k \in [1, N].$$

Remark 3.1. N must be a power of 2 for effective use of FFT.

Remark 3.2. m is either 1 or 2 for case 1 (see theorem 2.1).

Remark 3.3. K must be greater than m.

3.1. Adaptation of the fast algorithm for annular domains

The above algorithm can be easily modified to compute the singular integral operators within an annular region $\Omega_R = \{\sigma\colon\ R < |\sigma| < 1\}$ provided the annulus Ω_R is similarly discretized, i.e., with $N \times M$ lattice points with N equidistant points in the circular direction and M equidistant points in the radial direction. Since the origin is not part of the domain, the inner radius $r_1 = R \neq 0$. This requires minor modifications of the above algorithm which are rather straightforward because theorem 2.1 still applies on which the algorithm is based. Nonetheless, these are mentioned below to facilitate the implementation of the algorithm.

Algorithm 3.2. Algorithm 3.1 with the following minor modifications for the case when $h(\sigma)$ is a Hölder continuous function is referred to as algorithm 3.2 for our later purposes. In a follow-up paper [14], we apply this algorithm to quasiconformal mappings of doubly connected domains.

1. In Input, restrict $m = 1, 2$ and also specify R.
2. In step 1, change $r_1 = 0$ to $r_1 = R$.

Table 1
Computational complexity.

Step number	Operation count	Explanation
2	O($MN \ln N$)	Each FFT with N data points contributes $N \ln N$ operations. There are M such FFTs to be performed.
3	O(MN)	Computation of $C^{i,i+1}_{n,m}$, $i \in [1, M-1]$, contributes $2M$ operations for each fixed n. There are $2(K-m)$ such computations.
4	O(MN)	Computation of each $C_{n,m}(r_l)$ takes one operation. There are $2M(K-m)$ such computations.
5	O(MN)	Computation of each $B_{n,m}(r_l)$ takes 3 operations. There are $2M(K-m)$ such computations.
6	O(MN)	Computation of each $S_{n,m}(r_l)$ takes one operation. There are $2M(K-m)$ such computations.
7	O($MN \ln N$)	Computation of $T_m h(\sigma = r_l \mathrm{e}^{2\mathrm{i}\alpha_k})$, $k \in [1, N]$, for each fixed l by FFT contributes $N \ln N$ operations. There are M such FFTs to be performed.

3. In step 5, change $l \in [2, M]$ to $l \in [1, M]$.

4. In step 6, change $l \in [2, M]$ to $l \in [1, M]$. Since the origin is not part of the domain, we do not need to compute $S_{n,m}(0)$ in step 6.

3.2. *The algorithmic complexity*

We consider the computational complexity of the above algorithms. We discuss the asymptotic operation count, time complexity and storage requirement.

From table 1 we see that the asymptotic time complexity is O($MN \ln N$). The algorithm requires storage of the $2MK$ Fourier coefficients $h_n(r_l)$ in step 2, the $2M(K-m)$ Fourier coefficients $S_{n,m}(r_l)$ in step 6 and the MN values of $T_m h$ at MN grid points in step 7. Therefore, the asymptotic storage requirement is O(MN).

The algorithm is also inherently parallelizable and thus these estimates can be improved upon if the algorithm is implemented on a parallel machine which is a topic of further research.

4. Validation of the algorithm

In this section we validate the algorithm of section 3 as well as illustrate the algorithmic steps with two examples. Examples 4.1 and 4.2 validate the cases (i) and (ii) of theorem 2.1, respectively.

Example 4.1. Let $h(\zeta) = \bar{\zeta}$ and $m = 2$. Then

$$
\begin{aligned}
T_2h(\sigma) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{B(0;1)-B(\sigma;\varepsilon)} \frac{\bar\zeta}{(\zeta-\sigma)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= -\frac{1}{4\pi \mathrm{i}}\int_{\partial B(0;1)} \frac{\bar\zeta^2}{(\zeta-\sigma)^2}\,\mathrm{d}\zeta + \frac{1}{4\pi \mathrm{i}}\lim_{\varepsilon\to 0}\int_{\partial B(\sigma;\varepsilon)} \frac{\bar\zeta^2}{(\zeta-\sigma)^2}\,\mathrm{d}\zeta \\
&= -\frac{1}{4\pi \mathrm{i}}\int_{\partial B(0;1)} \frac{\bar\zeta^2}{(\zeta-\sigma)^2}\,\mathrm{d}\zeta + \frac{1}{4\pi \mathrm{i}}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{2\pi} \left(\bar\sigma + \varepsilon\,\mathrm{e}^{-\mathrm{i}\theta}\right)^2 \mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\theta \\
&= -\frac{1}{4\pi \mathrm{i}}\int_{\partial B(0;1)} \frac{\mathrm{d}\zeta}{\zeta^2(\zeta-\sigma)^2} = 0.
\end{aligned}
\tag{4.1}
$$

Below we apply the fast algorithm to evaluate the above integral T_2h at some grid points determined by the initialization.

Initialization: we take $M = 4$, $N = 32$ and $m = 2$.
Step 1. Set $K = N/8 = 4$.
Step 2. Compute the Fourier coefficients $h_n(r_l)$, $\forall l \in [1,4]$ and $n \in [-2,4]$:

$$h_n(r_l) = \begin{cases} 0 & \text{if } n \neq -1, \\ r_l & \text{if } n = -1. \end{cases}$$

Step 3. Compute $C_{n,2}^{i,i+1}$, $\forall i \in [1,3]$ and $n \in \{[-4,-2]\cup[0,2]\}$, using equation (2.31):

$$C_{n,2}^{i,i+1} = \begin{cases} 0 & \text{if } n \neq -3, \\ \dfrac{r_i^4 - r_{i+1}^4}{r_{i+1}^3} & \text{if } n = -3. \end{cases}$$

Step 4. Compute $C_{n,2}(r_l)$, $\forall l \in [1,4]$ and $n \in [-4,2]$, using corollaries 2.1–2.3:

$$C_{n,2}(r_l) = \begin{cases} 0 & \text{if } n \neq -3, \\ -r_l & \text{if } n = -3. \end{cases}$$

Step 5. Compute $B_{n,2}(r_l)$, $\forall l \in [2,4]$ and $n \in [-4,2]$, using equation (2.4):

$$B_{n,2}(r_l) = \begin{cases} 0 & \text{if } n \neq -3, \\ r_l & \text{if } n = -3. \end{cases}$$

Step 6. Compute $S_{n,2}(r_l)$, $\forall l \in [1,4]$ and $n \in [-4,2]$, using equation (2.1):

$$S_{n,2}(r_l) = 0.$$

Step 7. Compute $T_2h(r_l\mathrm{e}^{2\pi\mathrm{i}k/N})$, $\forall l \in [1,M]$ and $k \in [1,N]$, from its Fourier coefficients $S_{n,2}(r_l)$:

$$T_2h\left(r_l\mathrm{e}^{2\pi\mathrm{i}k/N}\right) = \sum_{n=-K}^{K-m} S_{n,2}(r_l)\,\mathrm{e}^{2\pi\mathrm{i}kn/N} = 0.$$

These values are in accordance with (4.1).

Example 4.2. Let $h(\zeta) = \zeta^4$ and $m = 3$, then

$$\begin{aligned}
T_3 h(\sigma) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{B(0;1)-B(\sigma;\varepsilon)} \frac{\zeta^4}{(\zeta-\sigma)^3}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= -\frac{1}{2\pi\mathrm{i}}\int_{\partial B(0;1)} \frac{\zeta^4\bar{\zeta}}{(\zeta-\sigma)^3}\,\mathrm{d}\zeta + \frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0}\int_{\partial B(\sigma;\varepsilon)} \frac{\zeta^4\bar{\zeta}}{(\zeta-\sigma)^3}\,\mathrm{d}\zeta \\
&= -\frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} \frac{\mathrm{i}\mathrm{e}^{4\mathrm{i}\theta}}{(\mathrm{e}^{\mathrm{i}\theta}-\sigma)^3}\,\mathrm{d}\theta \\
&\quad + \frac{1}{2\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_0^{2\pi} \left(\sigma+\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\right)^4\left(\bar{\sigma}\,\mathrm{e}^{-2\mathrm{i}\theta}+\varepsilon\,\mathrm{e}^{-3\mathrm{i}\theta}\right)\mathrm{d}\theta \\
&= -\frac{1}{2\pi\mathrm{i}}\int_{\partial B(0;1)} \frac{\zeta^3\,\mathrm{d}\zeta}{(\zeta-\sigma)^3} + 6\bar{\sigma}\sigma^2 \\
&= -3\sigma + 6\sigma^2\bar{\sigma} = \left(-3r+6r^3\right)\mathrm{e}^{\mathrm{i}\alpha}.
\end{aligned} \tag{4.2}$$

In order to compare (4.2) with the result of the fast algorithm, we apply the fast algorithm to evaluate the same integral at some grid points determined by the initialization.

Initialization: we take $M = 3$, $N = 32$ and $m = 3$.
Step 1. Set $K = N/8 = 4$.
Step 2. Compute the Fourier coefficients $h_n(r_l)$, $\forall l \in [1,3]$ and $n \in [-1,4]$:

$$h_n(r_l) = \begin{cases} 0 & \text{if } n \neq 4, \\ r_l^4 & \text{if } n = 4. \end{cases}$$

Step 3. Compute $C_{n,3}^{i,i+1}$, $i = 1,2$ and $n \in \{[-4,-3] \cup [0,1]\}$, using equation (2.31):

$$C_{n,3}^{i,i+1} = \begin{cases} 0, & (n,i) \neq (1,2), \\ 3\left(r_2 - r_2^3\right), & (n,i) = (1,2). \end{cases}$$

Step 4. Compute $C_{n,3}(r_l)$, $\forall l \in [1,3]$ and $n \in [-4,1]$, using corollaries 2.1–2.3:

$$C_{n,3}(r_l) = \begin{cases} 0, & (n,l) \neq (1,2), \\ 3\left(-r_2 + r_2^3\right) & (n,l) = (1,2). \end{cases}$$

Step 5. Compute $B_{n,3}(r_l)$, $l = 2,3$ and $n \in [-4,1]$, using equation (2.4):

$$B_{n,3}(r_l) = \begin{cases} 0, & n \neq 1, \\ 3r_l^3, & n = 1. \end{cases}$$

Step 6. Compute $S_{n,3}(r_l)$, $\forall l \in [1,3]$ and $n \in [-4,1]$, using equation (2.1):

$$S_{n,3}(r_l) = 0, \quad n \neq 1,$$

and

$$S_{1,3}(r_1) = 0, \qquad S_{1,3}(r_2) = -3r_2 + 6r_2^3, \quad \text{and} \quad S_{1,3}(r_3) = 3r_3^3.$$

Step 7. Compute $T_3h(r_l \mathrm{e}^{2\pi \mathrm{i}k/N})$, $\forall l \in [1,3]$ and $k \in [1,N]$, from its Fourier coefficients $S_{n,3}(r_l)$:

$$\begin{aligned} T_3h\big(r_1\mathrm{e}^{2\pi\mathrm{i}k/N}\big) &= 0, && \forall k \in [1,N], \\ T_3h\big(r_2\mathrm{e}^{2\pi\mathrm{i}k/N}\big) &= \big(-3r_2 + 6r_2^3\big)\,\mathrm{e}^{2\pi\mathrm{i}k/N}, && \forall k \in [1,N], \\ T_3h\big(r_3\mathrm{e}^{2\pi\mathrm{i}k/N}\big) &= 3r_3^3\,\mathrm{e}^{2\pi\mathrm{i}k/N}, && \forall k \in [1,N]. \end{aligned} \tag{4.3}$$

These values are in accordance with (4.2).

5. Conclusion

In this paper we have generalized a fast algorithm of Daripa [11] for rapid evaluation of some singular operators that may arise in many problems of pure and applied mathematics. In particular we have developed fast algorithms for evaluation of $T_m h(\sigma)$ for the following cases:

(i) The complex valued function $h(\sigma)$ is Hölder continuous with exponent γ, $0 < \gamma < 1$. In this case, $T_1h(\sigma)$ and $T_2h(\sigma)$ are defined as a Cauchy principal value and are evaluated by the fast algorithm described in section 3.

(ii) The complex valued function $h(\sigma)$ is analytic in the unit disk. In this case, $T_m h(\sigma)$ is defined for any finite positive integer m as a Cauchy principal value and is evaluated by the fast algorithm described in section 3.

A similar analysis can be carried out to develop fast and accurate algorithms for evaluating other singular and hypersingular integrals that arise in integral equation methods for solving partial differential equations. These ideas can also be extended to three dimensions and are currently in progress. A real challenge is to extend these algorithms to arbitrary geometries without losing accuracy. Applications of some of these algorithms to quasiconformal mappings can be found in [12,14]. Another application of these algorithms to efficient and accurate design of airfoils is in progress by the first author (PD) and will be reported in the future.

Acknowledgements

The first author's (PD) work has been supported by the Texas Advanced Research Program under Grant No. TARP-97010366-030, by the NSF under Grants No. DMS-9208061 and by the Office of the Vice President for Research and Graduate Studies at Texas A&M University. The allocation of computer resources by the Texas A&M Supercomputer Center is gratefully acknowledged. The authors also acknowledge the suggestions of the referees.

Appendix A

Proposition 1. For $\sigma \neq 0$,

$$-\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\zeta-\sigma)} = 0 \quad \text{and} \quad -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\zeta-\sigma)^2} = \frac{\bar{\sigma}}{\sigma}. \tag{A.1}$$

The result obtained here is used in proving lemmas 2.1 and 2.3.

Proof. We have

$$\begin{aligned}
&\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\zeta-\sigma)^2}\\
&\quad= \frac{1}{2\mathrm{i}}\int_{{}_{r-\varepsilon}\partial\Omega^*_{r+\varepsilon}} \frac{\bar{\zeta}}{(\zeta-\sigma)^2}\,\mathrm{d}\zeta\\
&\quad= \frac{1}{2\mathrm{i}}\left\{\int_{\partial B(0;r+\varepsilon)} \frac{\bar{\zeta}}{(\zeta-\sigma)^2}\,\mathrm{d}\zeta - \int_{\partial B(0;r-\varepsilon)} \frac{\bar{\zeta}}{(\zeta-\sigma)^2}\,\mathrm{d}\zeta - \int_{\partial B(\sigma;\varepsilon)} \frac{\bar{\zeta}}{(\zeta-\sigma)^2}\,\mathrm{d}\zeta\right\}\\
&\quad= \frac{1}{2\mathrm{i}}\left\{\int_{\partial B(0;r+\varepsilon)} \frac{(r+\varepsilon)^2}{\zeta(\zeta-\sigma)^2}\,\mathrm{d}\zeta - \int_{\partial B(0;r-\varepsilon)} \frac{(r-\varepsilon)^2}{\zeta(\zeta-\sigma)^2}\,\mathrm{d}\zeta\right.\\
&\qquad\left. - \int_0^{2\pi} \frac{(\bar{\sigma}+\varepsilon \mathrm{e}^{-\mathrm{i}\theta})}{\varepsilon^2\,\mathrm{e}^{2\mathrm{i}\theta}}\,\mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right\}\\
&\quad= \frac{1}{2\mathrm{i}}\left\{2\pi\mathrm{i}(r+\varepsilon)^2\left\{\frac{1}{\sigma^2}-\frac{1}{\sigma^2}\right\} - \frac{2\pi\mathrm{i}(r-\varepsilon)^2}{\sigma^2}\right\} = -\frac{\pi(r-\varepsilon)^2}{\sigma^2}.
\end{aligned} \tag{A.2}$$

Similarly it can be shown that

$$\begin{aligned}
\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\zeta-\sigma)} &= \frac{1}{2\mathrm{i}}\int_{{}_{r-\varepsilon}\partial\Omega^*_{r+\varepsilon}} \frac{\bar{\zeta}}{(\zeta-\sigma)}\,\mathrm{d}\zeta\\
&= \frac{1}{2\mathrm{i}}\left\{\int_{\partial B(0;r+\varepsilon)} \frac{\bar{\zeta}}{(\zeta-\sigma)}\,\mathrm{d}\zeta - \int_{\partial B(0;r-\varepsilon)} \frac{\bar{\zeta}}{(\zeta-\sigma)}\,\mathrm{d}\zeta\right.\\
&\qquad\left. - \int_{\partial B(\sigma;\varepsilon)} \frac{\bar{\zeta}}{(\zeta-\sigma)}\,\mathrm{d}\zeta\right\} = \frac{\pi(r-\varepsilon)^2}{\sigma} - \pi\bar{\sigma}.
\end{aligned} \tag{A.3}$$

Proposition 1 follows from (A.2) and (A.3). □

Proposition 2.

$$\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0. \tag{A.4}$$

The result obtained here is used in proving lemma 2.2.

Proof. Let $\delta > \varepsilon$ and $\Omega_\delta = {}_{r-\varepsilon}\Omega^*_{r+\varepsilon} \cap B(\sigma;\delta)$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad= \lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}\setminus\Omega_\delta} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta + \lim_{\varepsilon\to 0}\iint_{\Omega_\delta} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad\leqslant \delta^{\gamma-2}\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}\setminus\Omega_\delta} \mathrm{d}\xi\,\mathrm{d}\eta + \lim_{\varepsilon\to 0}\iint_{\Omega_\delta} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad= 0 + \lim_{\varepsilon\to 0}\iint_{\Omega_\delta} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad\leqslant \lim_{\varepsilon\to 0}\iint_{B(\sigma;\delta)-B(\sigma;\varepsilon)} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad= \lim_{\varepsilon\to 0}\int_\varepsilon^\delta\int_0^{2\pi} \rho^{\gamma-1}\,\mathrm{d}\theta\,\mathrm{d}\rho = \lim_{\varepsilon\to 0}\frac{2\pi}{\gamma}\big(\delta^\gamma-\varepsilon^\gamma\big) = \frac{2\pi}{\gamma}\delta^\gamma. \qquad \text{(A.5)}
\end{aligned}$$

Thus, for every fixed $\delta > \varepsilon$ we have

$$\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta \leqslant \frac{2\pi}{\gamma}\delta^\gamma. \qquad \text{(A.6)}$$

Therefore,

$$\lim_{\varepsilon\to 0}\iint_{{}_{r-\varepsilon}\Omega^*_{r+\varepsilon}} |\zeta-\sigma|^{\gamma-2}\,\mathrm{d}\xi\,\mathrm{d}\eta \leqslant \frac{2\pi}{\gamma}\lim_{\delta\to 0}\delta^\gamma, \qquad \text{(A.7)}$$

and since the integrand is positive, proposition 2 follows. □

Proposition 3. If $h(\zeta)$ is analytic in a neighborhood of the unit disk, then

$$\begin{aligned}
&\frac{\mathrm{d}^{m-1}}{\mathrm{d}\zeta^{m-1}}\left(\frac{h(\zeta)}{\zeta}\right)\bigg|_{\zeta=r\,\mathrm{e}^{\mathrm{i}\alpha}} \\
&\quad= \sum_{s=0}^{\infty}\frac{h_s(r)}{r^m}(s-1)(s-2)\cdots(s-m+1)\,\mathrm{e}^{(s-m)\mathrm{i}\alpha}. \qquad \text{(A.8)}
\end{aligned}$$

The result obtained here is used in proving lemma 2.6.

Proof. If $h(\zeta)$ is analytic in a neighborhood of the unit disk, then

$$h(\zeta) = \sum_{s=0}^{\infty}\frac{h_s(r)}{r^s}\zeta^s, \qquad \text{(A.9)}$$

where $h_s(r)$ is the sth Fourier series coefficient of $h(\zeta = r\,\mathrm{e}^{\mathrm{i}\theta})$. Notice that $h_s(r)/r^s$ for all $s > 0$ do not depend on r. Therefore

$$\begin{aligned}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\zeta^{m-1}}\left(\frac{h(\zeta)}{\zeta}\right) &= \sum_{s=0}^{\infty}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\zeta^{m-1}}\left(\frac{h_s(r)}{r^s}\zeta^{s-1}\right)\\ &= \sum_{s=0}^{\infty}\frac{h_s(r)}{r^s}(s-1)(s-2)\cdots(s-m+1)\zeta^{s-m}. \qquad \text{(A.10)}\end{aligned}$$

Proposition 3 follows on substituting $\zeta = r\,\mathrm{e}^{\mathrm{i}\alpha}$ in (A.10). □

Proposition 4. If

$$Q_{n,m}(r,\zeta) = \int_0^{2\pi}\frac{\mathrm{e}^{-\mathrm{i}n\alpha}}{(\zeta-\sigma)^m}\,\mathrm{d}\alpha, \quad \sigma = r\,\mathrm{e}^{\mathrm{i}\alpha},$$

then

$$Q_{n,m}(r,\zeta) = \begin{cases} 2(-1)^m\pi r^n\binom{-n-1}{m-1}\zeta^{-(m+n)}, & n \leqslant -m, |\zeta| < r,\\ 0, & n > -m, |\zeta| < r,\\ 0, & n < 0, |\zeta| > r,\\ 2\pi r^n\binom{m+n-1}{m-1}\zeta^{-(m+n)}, & n \geqslant 0, |\zeta| > r.\end{cases} \qquad \text{(A.11)}$$

The result obtained here is used in proving theorem 2.1.

Proof.

$$Q_{n,m}(r,\zeta) = \int_0^{2\pi}\frac{\mathrm{e}^{-\mathrm{i}n\alpha}}{(\zeta-\sigma)^m}\,\mathrm{d}\alpha = \mathrm{i}(-1)^{m+1}r^n\int_{\partial B(0;r)}\frac{\sigma^{-n}}{\sigma(\sigma-\zeta)^m}\,\mathrm{d}\sigma. \qquad \text{(A.12)}$$

Case 1: $n < 0, |\zeta| < r$.

$$\begin{aligned}Q_{n,m}(r,\zeta) &= (-1)^m r^n(-\mathrm{i})\frac{2\pi\mathrm{i}}{(m-1)!}\left\{\frac{\mathrm{d}^{m-1}}{\mathrm{d}\sigma^{m-1}}\left(\sigma^{-n-1}\right)\Big|_{\sigma=\zeta}\right\}\\ &= \left(\frac{(-1)^m r^n 2\pi}{(m-1)!}(-1)^{m-1}(n+1)(n+2)\cdots(n+m-1)\sigma^{-(n+m)}\right)\Bigg|_{\sigma=\zeta}\\ &= \frac{(-1)^m r^n 2\pi}{(m-1)!}(-n-1)(-n-2)\cdots(-n-m+1)\zeta^{-(n+m)}\\ &= \begin{cases} 2\pi r^n(-1)^m\binom{-n-1}{m-1}\zeta^{-(m+n)}, & n \leqslant -m,\\ 0, & -m-1 \leqslant n < 0.\end{cases} \qquad \text{(A.13)}\end{aligned}$$

Case 2: $n < 0, |\zeta| > r$.

$$Q_{n,m}(r,\zeta) = (-1)^m r^n \int_{\partial B(0;r)} \frac{\sigma^{-(n+1)}}{(\sigma-\zeta)^m}\,\mathrm{d}\sigma = 0. \tag{A.14}$$

Case 3: $n \geqslant 0, |\zeta| < r$.

$$\begin{aligned} Q_{n,m}(r,\zeta) &= (-1)^m r^n 2\pi \left\{ \frac{1}{(m-1)!} \cdot \frac{\mathrm{d}^{m-1}}{\mathrm{d}\sigma^{m-1}} \left(\sigma^{-(n+1)}\right)\Big|_{\sigma=\zeta} \right. \\ &\quad \left. + \frac{1}{n!} \cdot \frac{\mathrm{d}^n}{\mathrm{d}\sigma^n} \left(\frac{1}{(\sigma-\zeta)^m} \right)\Big|_{\sigma=0} \right\} \\ &= 2\pi r^n (-1)^m \left\{ \frac{(-1)^{m-1}(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} \zeta^{-(n+m)} \right. \\ &\quad \left. + (-1)^n m(m+1)\cdots\frac{(m+n-1)}{n!}\frac{1}{(-\zeta)^{m+n}} \right\} \\ &= 2\pi r^n (-1)^m \left\{ (-1)^{m-1} \frac{(n+m-1)!}{n!(m-1)!} \zeta^{-(n+m)} \right. \\ &\quad \left. + \frac{(-1)^m (m+n-1)!}{(m-1)!n!} \zeta^{-(n+m)} \right\} = 0. \end{aligned} \tag{A.15}$$

Case 4: $n \geqslant 0, |\zeta| > r$.

$$\begin{aligned} Q_{n,m}(r,\zeta) &= 2\pi r^n (-1)^m \left\{ \frac{1}{n!} \cdot \frac{\mathrm{d}^n}{\mathrm{d}\sigma^n} \left(\frac{1}{(\sigma-\zeta)^m} \right)\Big|_{\sigma=0} \right\} \\ &= \frac{2\pi r^n (-1)^{2m}}{n!} \cdot \frac{(m+n-1)!}{(m-1)!} \zeta^{-(m+n)} \\ &= \frac{2\pi r^n}{n!(m-1)!} (m+n-1)! \zeta^{-(m+n)} \\ &= 2\pi r^n \binom{m+n-1}{n} \zeta^{-(m+n)}. \end{aligned} \tag{A.16}$$

Proposition 4 follows from (A.13)–(A.16). □

Appendix B. Existence of the T_m-operator

Theorem B.1. If $h(\sigma)$ satisfies a Hölder condition with exponent γ, $0 < \gamma < 1$, and constant K in a neighborhood of ${}_0\Omega_1$, then T_1 and T_2 exists in the unit disk as a Cauchy principal value.

Proof. We define $T_m h(\sigma)$ as a Cauchy principal value, i.e.,

$$T_m h(\sigma) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \iint_{{}_0\Omega_1^*} \frac{h(\zeta)}{(\zeta - \sigma)^m} \, \mathrm{d}\xi \, \mathrm{d}\eta, \tag{B.1}$$

where

$${}_0\Omega_1^* = B(0;1) - B(\sigma;\varepsilon).$$

With this definition, it follows that

$$\begin{aligned}
\left|T_m h(\sigma)\right| &\leqslant \frac{1}{\pi} \lim_{\varepsilon \to 0} \left(\left| \iint_{{}_0\Omega_1^*} \frac{h(\zeta) - h(\sigma)}{(\zeta - \sigma)^m} \, \mathrm{d}\xi \, \mathrm{d}\eta \right| + \left| \iint_{{}_0\Omega_1^*} \frac{h(\sigma)}{(\zeta - \sigma)^m} \, \mathrm{d}\xi \, \mathrm{d}\eta \right| \right) \\
&\leqslant \frac{1}{\pi} \lim_{\varepsilon \to 0} \left(\iint_{{}_0\Omega_1^*} \frac{|h(\zeta) - h(\sigma)|}{|\zeta - \sigma|^m} \, \mathrm{d}\xi \, \mathrm{d}\eta + \left| \iint_{{}_0\Omega_1^*} \frac{h(\sigma)}{(\zeta - \sigma)^m} \, \mathrm{d}\xi \, \mathrm{d}\eta \right| \right) \\
&\leqslant \frac{1}{\pi} \lim_{\varepsilon \to 0} \left(\iint_{{}_0\Omega_1^*} K|\zeta - \sigma|^{\gamma - m} \, \mathrm{d}\xi \, \mathrm{d}\eta + \left| h(\sigma) \iint_{{}_0\Omega_1^*} \frac{1}{(\zeta - \sigma)^m} \, \mathrm{d}\xi \, \mathrm{d}\eta \right| \right) \\
&\leqslant \frac{1}{\pi} \lim_{\varepsilon \to 0} \left(\iint_{\varepsilon \leqslant |\zeta - \sigma| \leqslant R > 2} K|\zeta - \sigma|^{\gamma - m} \, \mathrm{d}\xi \, \mathrm{d}\eta \right. \\
&\quad \left. + \frac{|h(\sigma)|}{2} \left| \int_{\partial(B(0;1) - B(\sigma;\varepsilon))} \frac{\bar{\zeta}}{(\zeta - \sigma)^m} \, \mathrm{d}\zeta \right| \right) \\
&\leqslant \frac{1}{\pi} \lim_{\varepsilon \to 0} \left(K \int_{\varepsilon}^{R} \int_0^{2\pi} r^{\gamma - m + 1} \, \mathrm{d}\theta \, \mathrm{d}r \right. \\
&\quad \left. + \frac{|h(\sigma)|}{2} \left| \int_{\partial B(0;1)} \frac{\mathrm{d}\zeta}{\zeta(\zeta - \sigma)^m} - \mathrm{i}\varepsilon^{1-m} \int_0^{2\pi} \left(\bar{\sigma} + \varepsilon \, \mathrm{e}^{-\mathrm{i}\theta} \right) \mathrm{e}^{\mathrm{i}(1-m)\theta} \, \mathrm{d}\theta \right| \right) \\
&= \frac{2KR^{\gamma - m + 2}}{\gamma - m + 2} + \frac{|h(\sigma)|}{2\pi} \left| 0 - \begin{cases} 0 & \text{if } m = 2 \\ 2\pi\bar{\sigma} & \text{if } m = 1 \end{cases} \right| \\
&\leqslant \frac{2KR^{\gamma - m + 2}}{\gamma - m + 2} + \left|h(\sigma)\right| |\sigma|.
\end{aligned} \tag{B.2}$$

Since h is continuous in $B(0;1)$, it follows from (B.2) that

$$\left|T_m h(\sigma)\right| \leqslant \frac{2KR^{\gamma - m + 2}}{\gamma - m + 2} + \max_{\sigma \in B(0;1)} \left(\left|h(\sigma)\right| |\sigma| \right), \quad m = 1 \text{ or } 2 \text{ and } R > 2.$$

This completes the proof of theorem B.1. □

Theorem B.2. If $h(\sigma)$ is analytic in $B(0;1)$, then $T_m h(\sigma)$ exists in $B(0;1)$ for all finite positive integer m as a Cauchy principal value.

Proof. We define here as before

$$T_m h(\sigma) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\iint_{{}_0\Omega_1^*} \frac{h(\zeta)}{(\zeta-\sigma)^m}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{B.3}$$

Then it follows from using Green's theorem that

$$\begin{aligned} T_m h(\sigma) &= -\frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0}\int_{\partial(B(0;1)-B(\sigma;\varepsilon))} \frac{h(\zeta)\bar{\zeta}}{(\zeta-\sigma)^m}\,\mathrm{d}\zeta \\ &= -\frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0}\left\{\int_{\partial B(0;1)} \frac{h(\zeta)\bar{\zeta}}{(\zeta-\sigma)^m}\,\mathrm{d}\zeta - \int_{\partial B(\sigma;\varepsilon)} \frac{h(\zeta)\bar{\zeta}}{(\zeta-\sigma)^m}\,\mathrm{d}\zeta\right\} \\ &= -\frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\to 0}\left\{\int_{\partial B(0;1)} \frac{h(\zeta)}{\zeta(\zeta-\sigma)^m}\,\mathrm{d}\zeta \right. \\ &\quad \left. - \frac{\mathrm{i}}{\varepsilon^{m-1}}\int_0^{2\pi} h\big(\sigma+\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\big)\big(\bar{\sigma}\,\mathrm{e}^{\mathrm{i}(1-m)\theta}+\varepsilon\,\mathrm{e}^{-\mathrm{i}m\theta}\big)\,\mathrm{d}\theta\right\}. \end{aligned} \tag{B.4}$$

Because of analyticity of function h, we can write

$$h\big(\sigma+\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\big) = \sum_{n=0}^{\infty} a_n\varepsilon^n\,\mathrm{e}^{\mathrm{i}n\theta}, \tag{B.5}$$

where a_n's will depend on σ. Substituting (B.5) into (B.4), we have

$$T_m h(\sigma) = -(b_1+b_2) - \lim_{\varepsilon\to 0}\frac{\mathrm{i}}{\varepsilon^{m-1}}\int_0^{2\pi}\sum_{n=0}^{\infty} a_n\varepsilon^n\,\mathrm{e}^{\mathrm{i}n\theta}\big(\bar{\sigma}\,\mathrm{e}^{\mathrm{i}(1-m)\theta}+\varepsilon\,\mathrm{e}^{-\mathrm{i}m\theta}\big)\,\mathrm{d}\theta, \tag{B.6}$$

where b_1 and b_2 are the residues of $h(\zeta)/(\zeta(\zeta-\sigma)^m)$ at 0 and σ of order 1 and m, respectively. Therefore,

$$\begin{aligned} T_m h(\sigma) &= -(b_1+b_2) - \lim_{\varepsilon\to 0}\frac{\mathrm{i}}{\varepsilon^{m-1}}\sum_{n=0}^{\infty}\bar{\sigma}a_n\varepsilon^n\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n+1-m)\theta}\,\mathrm{d}\theta \\ &\quad - \lim_{\varepsilon\to 0}\frac{\mathrm{i}}{\varepsilon^{m-1}}\sum_{n=0}^{\infty} a_n\varepsilon^{n+1}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta \\ &= -(b_1+b_2) - \lim_{\varepsilon\to 0}\left\{\frac{2\pi\mathrm{i}}{\varepsilon^{m-1}}\bar{\sigma}a_{m-1}\varepsilon^{m-1} - \frac{2\pi\mathrm{i}}{\varepsilon^{m-1}}a_m\varepsilon^{m+1}\right\} \\ &= -(b_1+b_2) - 2\pi\mathrm{i}\lim_{\varepsilon\to 0}\big\{\bar{\sigma}a_{m-1} - a_m\varepsilon^2\big\} \\ &= -(b_1+b_2) - 2\pi\mathrm{i}\bar{\sigma}a_{m-1}. \end{aligned} \tag{B.7}$$

This completes the proof of theorem B.2. □

References

[1] L.V. Ahlfors, *Lectures in Quasiconformal Mappings* (Van Nostrand Reinhold, New York, 1966).
[2] S.R. Bell, *The Cauchy Transform, Potential Theory and Conformal Mapping* (CRC Press, New York, 1992).
[3] L. Bers, *Mathematical Aspects of Subcritical and Transonic Gas Dynamics* (Wiley, New York, 1958).
[4] L. Bers, Quasiconformal mappings, with applications to differential equations, function theory and topology, Amer. Math. Soc. Bull. 83(6) (1977) 1083–1100.
[5] L. Bers and L. Nirenberg, On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications, in: *Convegno Internazionale Suelle Equaziono Cremeonese*, Roma (1955) pp. 111–140.
[6] L. Bers and L. Nirenberg, On linear and nonlinear elliptic boundary value problems in the plane, in: *Convegno Internazionale Suelle Equaziono Cremeonese*, Roma (1955) pp. 141–167.
[7] B. Bojarski and T. Iwanier, Quasiconformal mappings and nonlinear elliptic equations in two variables, I and II, Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys. 12 (1974) 473–478 and 479–484.
[8] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II (Wiley, New York, 1961).
[9] P. Daripa, On applications of a complex variable method in compressible flows, J. Comput. Phys. 88(2) (1990) 337–361.
[10] P. Daripa, On a numerical method for quasiconformal grid generation, J. Comput. Phys. 96 (1991) 229–236.
[11] P. Daripa, A fast algorithm to solve nonhomogeneous Cauchy–Riemann equations in the complex plane, SIAM J. Sci. Statist. Comput. 13(6) (1992) 1418–1432.
[12] P. Daripa, A fast algorithm to solve the Beltrami equation with applications to quasiconformal mappings, J. Comput. Phys. 106 (1993) 355–365.
[13] P. Daripa, Generalization of a fast algorithm, preprint (1993).
[14] P. Daripa and D. Mashat, An efficient and novel numerical method for quasiconformal mappings of doubly connected domains, Numer. Algorithms, this issue.
[15] M. Gonzales, *Classical Complex Analysis* (Marcel Dekker, New York, 1992).
[16] S. Krushkal, *Quasiconformal Mapping and Riemann Surfaces* (Cambridge Univ. Press, New York, 1961).
[17] J. Lawrynowicz, *Quasiconformal Mappings in the Plane*, Lecture Notes in Mathematics 978 (Springer, Berlin, 1983).
[18] C.B. Morrey, On the solutions of quasi-linear elliptic differential equations, Trans. Amer. Math. Soc. 43 (1938) 126–166.
[19] A. Mshimba and W. Tutschke, *Functional Analytic Methods in Complex Analysis and Applications to Partial Differential Equations* (World Scientific, Singapore, 1990).
[20] L. Nirenberg, On nonlinear elliptic differential equations and Hölder continuity, Comm. Pure Appl. Math. 6 (1953) 103–156.