Physica A 390 (2011) 3069–3076

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

On estimates for short wave stability and long wave instability in three-layer Hele-Shaw flows

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ARTICLE INFO

Article history:
Received 21 November 2010
Received in revised form 19 April 2011
Available online 2 May 2011

Keywords:
Hele-Shaw flows
Upper bound
Lower bound
Growth rates
Linear stability

ABSTRACT

We consider the linear stability of three-layer Hele-Shaw flows with each layer having constant viscosity and viscosity increasing in the direction of a basic uniform flow. While the upper bound results on the growth rate of long waves are well known from our earlier works, lower bound results on the growth rate of short stable waves are not known to date. In this paper, we obtain such a lower bound. In particular, we show the following results: (i) the lower bound for stable short waves is also a lower bound for all stable waves, and the exact dispersion curve for the most stable eigenvalue intersects the dispersion curve based on the lower bound at a wavenumber where the most stable eigenvalue is zero; (ii) the upper bound for unstable long waves is also an upper bound for all unstable waves, and the exact dispersion curve for the most unstable eigenvalue intersects the dispersion curve based on the upper bound at a wavenumber where the most unstable eigenvalue is zero. Numerical results are provided which support these findings.

1. Introduction

The depth-averaged velocity of a fluid flowing through the gap in a Hele-Shaw cell resembles the formula for Darcy's law, which is applicable to porous media flows. The viscous profile created due to rarefaction waves behind a sweeping front in two-phase immiscible flows in porous media can be modeled using a viscous profile behind the sweeping front in a Hele-Shaw flow. These analogies have motivated extensive studies in two-layer Hele-Shaw flows (see [1–4]) to understand various issues related to porous media flows. The design of chemical enhanced oil recovery (EOR) processes usually involves flooding oil reservoirs with a sequence of displacing fluids of various compositions containing chemicals (see [5]). A closely analogous system is that of multi-layer Hele-Shaw flows, which have recently been studied by Daripa [6]. This same analogy motivates the current study of three-layer Hele-Shaw flows in order to gain an understanding of some of the complicated issues surrounding EOR technology.

The three-layer Hele-Shaw model consists of three different fluid phases in three distinct regions separated by sharp interfaces which have interfacial tensions. It is worth mentioning that the role of interfacial tension in actual porous media is more involved, which results in diffuse two-phase regions, not sharp interfaces (see [7]). Towards this end, we mention that the linear stability of miscible displacement processes in porous media in the absence of dispersion has been studied earlier (see [8]). The approximation of diffused interfaces by sharp interfaces in our Hele-Shaw model allows exact studies through analysis of some hydrodynamic stability issues which play important roles in enhanced oil recovery. Many such issues related to three-layer Hele-Shaw flows have been studied by the author and his collaborators in recent years (see [6,9–11]). In such flows as our present interest in this paper, there is a middle layer of fluid of constant viscosity μ between the displacing fluid of viscosity μ_l and the displaced fluid of viscosity μ_r. The viscosity μ is chosen so that

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doi:10.1016/j.physa.2011.04.028

$\mu_l < \mu < \mu_r$. Two initially planar interfaces including all three fluids in the three layers move with velocity U along the positive direction of x-axis. The y-axis is in the plane of the plates and extends all the way to infinity in both directions of the y-axis. In a frame moving with velocity U, $x = 0$ and $x = -L$ are taken to be initial locations of the two planar interfaces, with the displaced fluid extending all the way to $x = \infty$ and the displacing fluid extending all the way to -∞. The interfacial tension at the leading interface at $x = 0$ is denoted by T and that at the trailing interface at $x = -L$ is denoted by S.

The eigenvalue problem arising from linear stability analysis of this uniform flow using equations relevant for Hele-Shaw flows has been derived in Daripa [11] and also in some references cited therein. This derivation is outlined here briefly. The disturbances $(\tilde{u}, \tilde{v}, \tilde{p})$ in basic velocity $(U, 0)$ and basic pressure P (see [11] for P) are first decomposed in normal modes according to the ansatz

$$(\tilde{u}, \tilde{v}, \tilde{p}) = (f(x), \phi(x), \psi(x))\mathrm{e}^{(\mathrm{i}ky+\sigma t)}, \tag{1}$$

where k is the wavenumber and σ is the growth rate. Then these are used in the linearized disturbance equations arising from Hele-Shaw flow equations and the linearized dynamic and kinematic boundary conditions. The resulting equations are then manipulated to obtain the following eigenvalue problem in $f(x)$. Details of this derivation can be found in [11].

$$f_{xx} - k^2 f = 0, \tag{2}$$

$$f_x^+(-L) = (\lambda r + s)f(-L), \qquad f_x^-(0) = (\lambda p + q)f(0), \tag{3}$$

where $\lambda = 1/\sigma$ and r, s, p, and q are given by

$$r = \{(\mu_l - \mu)Uk^2 + S k^4\}/\mu, \qquad s = \mu_l k/\mu \geq 0, \tag{4}$$

$$p = \{(\mu_r - \mu)Uk^2 - T k^4\}/\mu, \qquad q = -\mu_r k/\mu \leq 0. \tag{5}$$

Notice that the eigenvalue σ appears in the boundary conditions (3) through λ. There are two non-trivial eigenvalues $\sigma_+(k)$ and $\sigma_-(k)$ (where $\sigma_+(k) > \sigma_-(k)$) of this eigenvalue problem; this has been discussed in [11].

Past works on this problem that are relevant for this paper are reviewed briefly here. This is also necessary for the purpose of continuity so that we place the contribution of this paper in a proper perspective. An *absolute* upper bound on the growth rate has been derived in [9,10] in two different ways. In [9], this has been done using numerical analysis of the discrete version of the above eigenvalue problem followed by an application of Gerschgorin's localization theorem for eigenvalues. Since an absolute upper bound need not be the best upper bound (i.e., maximum growth rate), we sought to derive this by another approach, hoping that an improved upper bound can be obtained. In a subsequent paper [10], this was done using the variational formulation of the eigenvalue problem, which is more elegant and straightforward. Even though we have not emphasized the *local* upper bound on the growth rates of long waves in these two papers, they are embedded in the content of those papers from which the *local* upper bound result for long waves follows. However, to date, no local lower bound result on the growth rates of short waves exists. This is partly due to the fact that such short waves are stable due to surface tension effects. Therefore, it was felt at the time that the *local* lower bound for short waves may not be of interest. In retrospect, it turns out that this is not true, for many reasons. Short waves participate and thus play an important role in determining the overall stability in an experimental set up of finite (in the y-direction) width of the plates. More importantly, in this paper we obtain stronger results: the local upper bound on the growth rates for most unstable modes which include the long waves, and the local lower bound on the growth rates for most stable modes which include the short waves.

For our purposes below, we recall from [6] that σ is referred to as the growth rate even when $\sigma < 0$. The growth rate ($\sigma < 0$) characteristic of any short wave depends on the values of the parameters such as the viscosity of the three fluids, the interfacial tensions, and the length of the middle layer. Therefore, the growth rate of a short wave can vary widely in the space of these parameters. Even for a fixed set of parameter values, the growth rate can decrease rapidly with increasing wavenumber. If a local lower bound on the growth rate for short waves also shares these same properties of the growth rate with respect to variation in one or more of these parameters, then it is possible to use the local lower bound to find approximately the qualitative effect of changes in parameter values on the stability of short waves. This quantitative information can be useful in the selection of one or more of the parameters appropriately in order to achieve some stabilization objectives of these short waves and in particular the system as a whole. In fact, its effect on the size of the unstable band (which usually is outside the band of short waves) can also be inferred in a qualitative sense, i.e., whether more or fewer unstable waves participate in determining the stability of the system as the parameter values are changed. For example, if it is found from the *local* lower bound that the short waves as a group can become more stable as a result of some changes in some parameter values, then it is very likely that the unstable band will also shrink in size.

Below, we first analyze the above eigenvalue problem to estimate this local lower bound for short waves. This can be done in three different ways, all leading to the same result, and two of these three methods parallel the ones we have presented in [9,10] for estimating the *local* upper bound for long waves. Below, we present all these three approaches to derive some inequalities from which not only the local lower bound on the growth rate for short waves but also the local upper bound on the growth rate of long waves follows. Some degree of overlap with the author's work in [9,10] is not only unavoidable but also necessary in order to establish the equivalence among these three methods, one of which is new. In this paper, we also compare these bounds with the exact growth rates of these waves found numerically in [6]. Such comparisons validate not only the bounds but also some other results derived below.

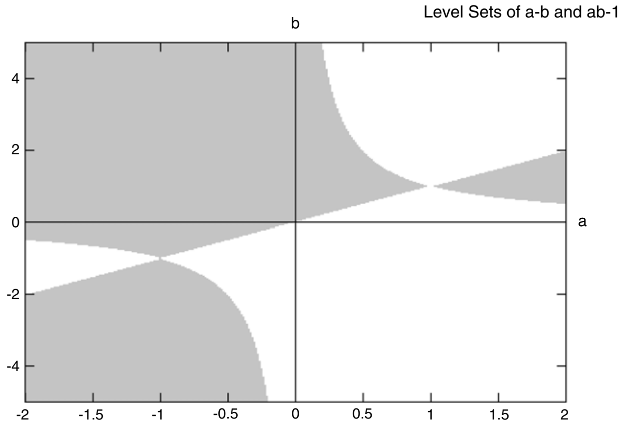


Fig. 1. Plots of functions $a - b = 0$ and $ab - k^2 = 0$ for $k = 1$ with a and b as the two axes. The regions where both $a - b > 0$ and $ab - k^2 > 0$ hold or both $a - b < 0$ and $ab - k^2 < 0$ hold are shaded.

2. The estimates of the growth rate σ

Upper and lower bounds on the growth rates of short and long waves are derived in this section. We will later show that these are also the bounds on the growth rates for unstable and stable modes. This derivation will be based on the following proposition, which we prove first. Recall that $\lambda = 1/\sigma$.

Proposition 1. *There are only the following two possibilities.*

$$a := \lambda p + q > 0, \quad \text{or} \quad b := \lambda r + s < 0. \tag{6}$$

Proof. The above result will be proved using three methods: (i) calculation using the general solution of (2); (ii) discretization of the stability system (2)–(3) and Gerschgorin's localization theorem for eigenvalues; and (iii) variational formulation of the stability system.

(i) Calculation using the general solution of (2). The general solution of system (2)–(3) is given by

$$f(x) = A\,\mathrm{e}^{kx} + B\,\mathrm{e}^{-kx}, \tag{7}$$

where A and B satisfy the algebraic system

$$\left.\begin{array}{l} k(A - B) = (\lambda p + q)(A + B), \\ k(A\,\mathrm{e}^{-kL} - B\,\mathrm{e}^{kL}) = (\lambda r + s)(A\,\mathrm{e}^{-kL} + B\,\mathrm{e}^{kL}). \end{array}\right\} \tag{8}$$

The above system has non-trivial solution A, B iff the determinant is zero; that is,

$$(a - k)\,\mathrm{e}^{kL}(b + k) - (a + k)\,\mathrm{e}^{-kL}(b - k) = 0, \tag{9}$$

or

$$(ab - k^2)(\mathrm{e}^{2kL} - 1) + k(a - b)(\mathrm{e}^{2kL} + 1) = 0. \tag{10}$$

In [6], this algebraic equation has been solved for eigenvalues σ_+ and σ_- as a function of wavenumber k. Suppose that $a < 0$ and $b > 0$. Then both the terms in the left-hand side of the Eq. (10) are negative, and the sum cannot equate to the right-hand side, which is zero. One can easily verify this and other possible scenarios from Fig. 1 and arrive at the conclusion (6). This figure shows zero-level sets of $(ab - k^2)$ and $(a - b)$ using a and b as the two axes. The five curves in the figure then clearly identify regions when both $ab > k^2$ and $a > b$ hold or when both $ab < k^2$ and $a < b$ hold. These regions are shown shaded in the figure. These are then the regions in which the solution (a, b) of (10) cannot lie, thus justifying the assertion (6).

(ii) Discretization of the stability system (2)–(3) and Gerschgorin's localization theorem for eigenvalues. The discretization version is first recalled here from [9]. The domain $[-L, 0]$ is discretized into M subintervals of equal length $d = L/M$ by introducing the points $x_0 = 0, x_1 = -d, \ldots, x_i = -id, \ldots, x_M = -L$. The notation $f(x_i) = f_i$ is used. The derivative f_{xx} at the interior points $x_1, x_2, \ldots, x_{M-1}$ has been approximated as follows.

$$f_{xx}(x_i) \approx \frac{f(x_i + d) - 2f(x_i) + f(x_i - d)}{d^2}. \tag{11}$$

The end-point derivatives $f_x^-(x_0)$ and $f_x^+(x_M)$ are approximated as follows.

$$f_x(x_0) \approx \frac{f_0 - f_1}{d}, \qquad f_x(x_M) \approx \frac{f_{M-1} - f_M}{d}. \tag{12}$$

Therefore the boundary conditions (3) become

$$(f_0 - f_1)/d = (\lambda p + q) f_0, \qquad (f_{M-1} - f_M)/d = (\lambda r + s) f_M. \tag{13}$$

The discretized form of the stability system (2)–(3) is given by

$$E_{ij} f_j = F_{ij} f_j, \tag{14}$$

where E is the tridiagonal matrix with its entries given by

$$\begin{aligned} &E_{jj} = -2 \quad \text{except } E_{0,0} = (1 - d\,q) \quad \text{and} \quad E_{M,M} = -(1 + ds), \\ &E_{j-1,j} = E_{j,j+1} = 1 \quad \text{except } E_{0,1} = -1, \end{aligned} \tag{15}$$

and F is the diagonal matrix with its entries given by

$$F_{i,i} = d^2 k^2 \quad \text{except } F_{0,0} = \lambda d\,p \quad \text{and} \quad F_{M,M} = \lambda d\,r. \tag{16}$$

In the particular case $M = 4$, we have three equidistant interior points, and system (14) becomes

$$\left.\begin{aligned} &f_0(1 - d\,q) - f_1 = (\lambda d\,p) f_0, \\ &f_0 - 2f_1 + f_2 = d^2 k^2 f_1, \\ &f_1 - 2f_2 + f_3 = d^2 k^2 f_2, \\ &f_2 - 2f_3 + f_4 = d^2 k^2 f_3, \\ &f_3 - f_4(1 + d\,s) = (\lambda d\,r) f_4. \end{aligned}\right\} \tag{17}$$

Using Gerschgorin's theorem, we obtain, from (14),

$$|E_{kk} - F_{kk}| \le \sum_{j \ne k} |E_{kj} f_j| / |f_k| \le \sum_{j \ne k} |E_{kj}|,$$

if $\max |f_i| = |f_k|$. Now, the following three possibilities exist.
(a) If $\max |f_i| = |f_j|, \ 0 < j < M$, then, from (14), we obtain

$$|d^2 k^2 + 2| \le 2 \Rightarrow -4 \le d^2 k^2 \le 0. \tag{18}$$

It is obvious that this last relation is false.
(b) If $\max |f_i| = |f_0|$, then we obtain

$$|\lambda d\,p - 1 + d\,q| \le 1 \Rightarrow 0 \le d(\lambda p + q) \le 2. \tag{19}$$

(c) If $\max |f_i| = |f_M|$, then we obtain

$$|\lambda dr + 1 + d\,s| \le 1 \Rightarrow -2 \le d(\lambda r + s) \le 0. \tag{20}$$

Thus we see that only the possibilities (b) and (c) are meaningful, from which we again get the relation (6).

(iii) Variational formulation of the stability system. Multiplying Eq. (2) by $f(x)$ and then integrating the resulting equation in the interval $[-L, 0]$, we obtain, after using the boundary conditions (3),

$$(\lambda p + q) f^2(0) - (\lambda r + s) f^2(-L) = \int_{-L}^{0} f_x^2 \, dx + k^2 \int_{-L}^{0} f^2 \, dx. \tag{21}$$

Therefore, using the notations introduced in (6), we have

$$a f^2(0) - b f^2(-L) \ge 0. \tag{22}$$

Suppose that $a < 0$ and $b > 0$; then both terms in the above inequality are negative, and the sum cannot be positive. Therefore, the assumption that $a < 0$ and $b > 0$ is false, and we obtain the result (6).

This completes the proof of the Proposition 1 in three different ways. □

We see from (5) that $p > 0$ if $0 < k < k_1$ and $p < 0$ if $k > k_1$, where $k_1^2 = U(\mu_r - \mu)/T$. Similarly, from (4), $(-r) > 0$ if $0 < k < k_2$ and $(-r) < 0$ if $k > k_2$, where $k_2^2 = U(\mu - \mu_l)/S$. Therefore, both $p > 0$ and $(-r) > 0$ hold when $0 < k < \min(k_1, k_2)$ (which we call "small" wavenumber below) and both $p < 0$ and $(-r) < 0$ hold when $k > \max(k_1, k_2)$ (which we call "large" wavenumber below). In the following, we analyze these two cases: the case of "small" wavenumbers for which $p > 0$ and $(-r) > 0$, and the case of "large" wavenumbers for which $p < 0$ and $(-r) < 0$. We can see that (recall $\mu_l < \mu < \mu_r$)

$$p > 0 \quad \text{and} \quad (-r) > 0 \Leftrightarrow k^2 \le \min\left\{\frac{U(\mu_r-\mu)}{T}, \frac{U(\mu-\mu_l)}{S}\right\}, \tag{23}$$

$$p < 0 \quad \text{and} \quad (-r) < 0 \Leftrightarrow k^2 \ge \max\left\{\frac{U(\mu_r-\mu)}{T}, \frac{U(\mu-\mu_l)}{S}\right\}. \tag{24}$$

We first consider the case of small wavenumbers. Then, using (23) and Proposition 1, we obtain the following two possibilities.

$$\lambda > \frac{-q}{p} > 0 \Rightarrow \lambda > 0 \quad \text{and} \quad \sigma < \frac{p}{-q} = \frac{Uk(\mu_r-\mu)-k^3T}{\mu_r}, \tag{25}$$

or

$$\lambda > \frac{s}{-r} > 0 \Rightarrow \lambda > 0 \quad \text{and} \quad \sigma < \frac{-r}{s} = \frac{Uk(\mu-\mu_l)-k^3S}{\mu_l}. \tag{26}$$

Therefore, in this case, we obtain the following upper bound $\sigma_{\rm ul}$ on the growth rate of long waves (i.e., small wavenumbers).

$$0 < \sigma < \sigma_{\rm ul} = \max\left\{\frac{Uk(\mu_r-\mu)-k^3T}{\mu_r}, \frac{Uk(\mu-\mu_l)-k^3S}{\mu_l}\right\}. \tag{27}$$

This upper bound for long waves is consistent with one of our results in [6]. There we have shown that $\sigma_{\rm ul}$ is an upper bound for all unstable waves (the word unstable was inadvertently left out from the third line after Eq. (33) in [6]), i.e. for all waves in the range

$$k \le \max(k_1, k_2) = \max\left\{\sqrt{\frac{U(\mu_r-\mu)}{T}}, \sqrt{\frac{U(\mu-\mu_l)}{S}}\right\}. \tag{28}$$

This range contains all long waves, and thus the upper bound result (27) for long waves is consistent with our result in [6]. It is easy to verify that $\sigma_{ul} = 0$ at $k = \max(k_1, k_2)$. We have also shown in [6] that $\sigma^+ = 0$ at $k = \max(k_1, k_2)$. Thus plots of σ_{ul} versus wavenumber k should intersect the dispersion curves $\sigma^+(k)$ at $k = \max(k_1, k_2)$, where $\sigma_{ul} = \sigma^+ = 0$. This, along with the upper bound result (27), will be validated below numerically.

In the case of large wavenumbers for which (24) holds, the inequality signs in the relations (25)–(26) are reversed because p, $(-r)$ are negative. Therefore, the growth rate of short waves (i.e., large wavenumbers) becomes negative, and we obtain the following lower bound $\sigma_{\rm ls}$ for short waves.

$$0 > \sigma > \sigma_{\rm ls} = \min\left\{\frac{Uk(\mu_r-\mu)-k^3T}{\mu_r}, \frac{Uk(\mu-\mu_l)-k^3S}{\mu_l}\right\}. \tag{29}$$

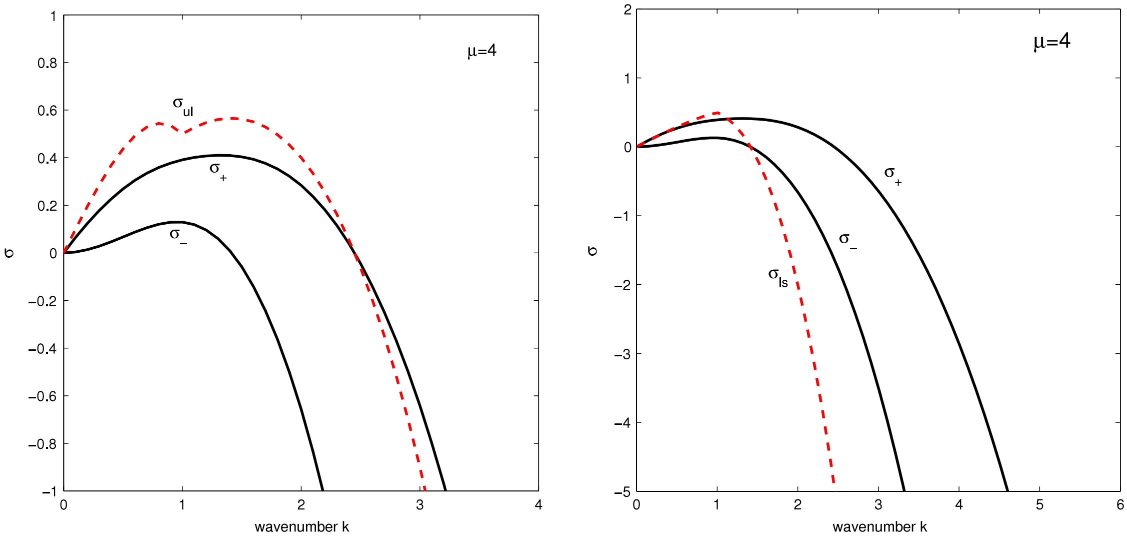
The new principal element of this paper is the last estimate (29) for short waves. It is also easy to verify that $\sigma_{ls} = 0$ at $k = \min(k_1, k_2)$. We have also shown in [6] that $\sigma^- = 0$ at $k = \min(k_1, k_2)$. This proves that the plot of σ_{ls} versus wavenumber k will intersect the dispersion curve $\sigma^-(k)$ at $k = \min(k_1, k_2)$, where $\sigma_{ls} = \sigma^- = 0$. This, along with the lower bound result (29), is validated numerically in the next section.

3. Numerical results

We have obtained the above upper and lower bounds on the growth rates of short and long waves, respectively, but it turns out, as explained and justified above, that these bounds also hold for all unstable and stable waves, respectively. Since, in general, the bounds are rarely optimal, allowing room for possible further improvement in these estimates through some different kind of analysis which we are not aware of at this point, it is useful to test the tightness of these bounds against exact calculations of the dispersion curves. In [6], such dispersion curves have been obtained numerically. The upper and lower bounds given above will now be compared with such numerically obtained exact dispersion curves.

In Figs. 2 through 5, we present plots for several choices of the set (S, T, U, L, μ) with the viscosities of the end layers fixed, namely $\mu_r = 10$ and $\mu_l = 2$. Fig. 2(a) shows the plots of σ_+, σ_- & $\sigma_{\rm ul}$ versus k and Fig. 2(b) shows the plots of σ_+, σ_- & $\sigma_{\rm ls}$ versus k for one such choice of the set: $(S, T, U, L, \mu) = (1, 1, 1, 1, 4)$. For the other three choices of the set (S, T, U, L, μ), such plots are shown in Figs. 3–5 for $(S, T, U, L, \mu) = (1, 1, 1, 0.5, 4)$, $(S, T, U, L, \mu) = (1, 0.5, 1, 1, 4)$, and $(S, T, U, L, \mu) = (0.5, 1, 1, 1, 4)$, respectively. These figures support the validity of the estimates $\sigma_{\rm ul}$ and $\sigma_{\rm ls}$. Also, these figures support our results: $\sigma_{ul} = \sigma^+ = 0$ at $k = \max(k_1, k_2)$ and $\sigma_{ls} = \sigma^- = 0$ at $k = \min(k_1, k_2)$. The validity of these has been confirmed for many other choices of the set (S, T, U, L, μ).

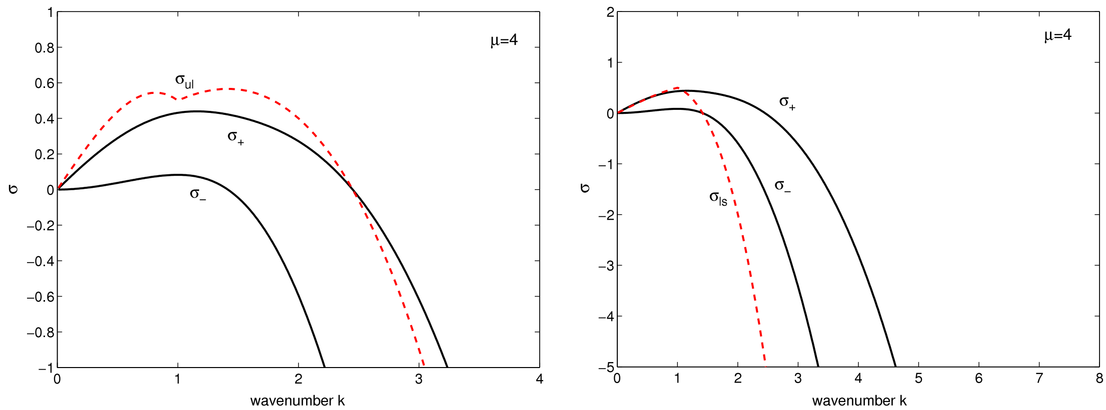
The point of illustrating four typical cases is to have a qualitative idea of the trend of the difference between the exact values of the growth rates and the corresponding bounds as the parameters are varied. Giving more case studies than one



(a) Validation of the upper bound σ_{ul} for long waves.

(b) Validation of the lower bound σ_{ls} for short waves.

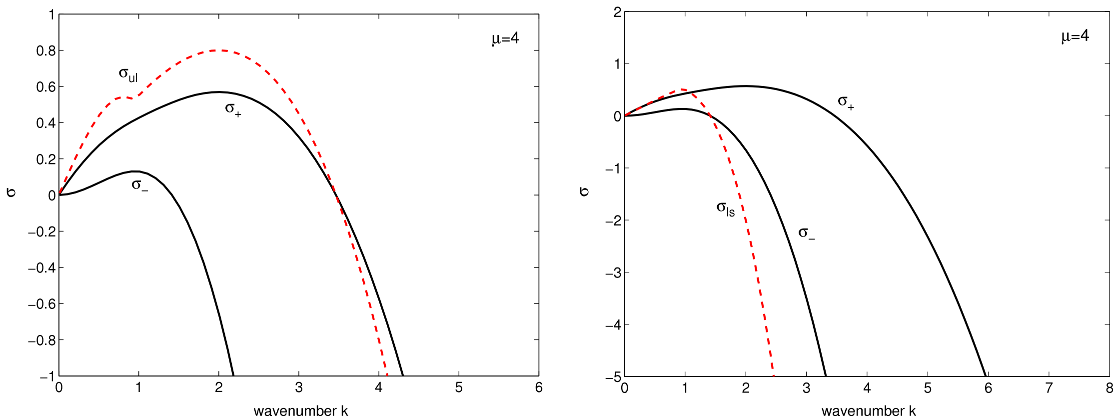
Fig. 2. Plots of σ_+, σ_-, & σ_{ul} versus k in the left subfigure, (a). Similarly, plots of σ_+, σ_-, & σ_{ls} versus k in the right subfigure, (b). The parameter values are $S = T = U = 1$, $L = 1$, and $\mu = 4$.



(a) Validation of the upper bound σ_{ul} for long waves.

(b) Validation of the lower bound σ_{ls} for short waves.

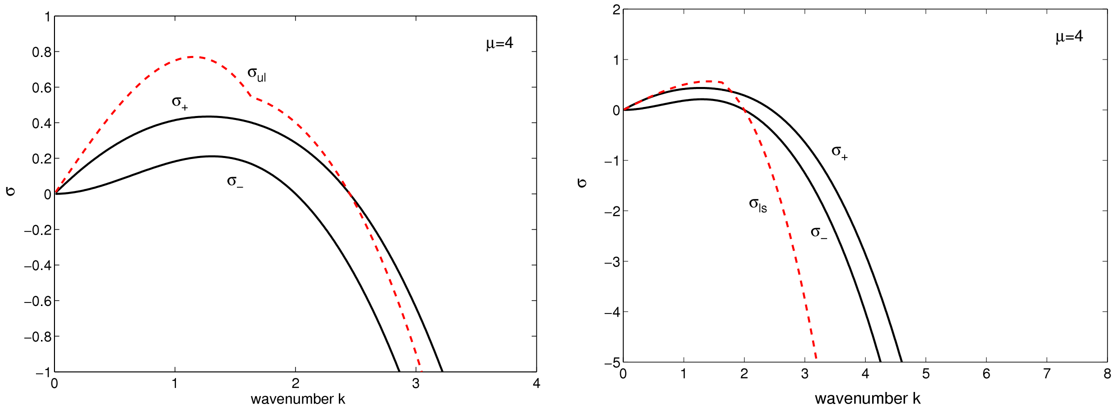
Fig. 3. Plots of σ_+, σ_-, & σ_{ul} versus k in the left subfigure, (a). Similarly, plots of σ_+, σ_-, & σ_{ls} versus k in the right subfigure, (b). The parameter values are $S = T = U = 1$, $L = 0.5$, and $\mu = 4$.



(a) Validation of upper bound σ_{ul} for long waves.

(b) Validation of the lower bound σ_{ls} for short waves.

Fig. 4. Plots of σ_+, σ_-, & σ_{ul} versus k in the left subfigure, (a). Similarly, plots of σ_+, σ_-, & σ_{ls} versus k in the right subfigure, (b). The parameter values are $S = L = U = 1$, $T = 0.5$, and $\mu = 4$.



(a) Validation of upper bound σ_{ul} for long waves.

(b) Validation of the lower bound σ_{ls} for short waves.

Fig. 5. Plots of σ_+, σ_-, & σ_{ul} versus k in the left subfigure, (a). Similarly, plots of σ_+, σ_-, & σ_{ls} versus k in the right subfigure, (b). The parameter values are $L = T = U = 1$, $S = 0.5$, and $\mu = 4$.

usually tends to answer the questions that may otherwise arise in readers' minds. For example, from these typical case studies and the associated figures, one finds that the plots of the upper bound σ_{ul} versus k in regions of interest (long wave regime) can have a double-hump or a single-hump characteristic (one being more pronounced than the other when there is a double hump). In Figs 2(a), 3(a), 4(a), and 5(a), the upper bound plots shown by the dashed curves are not that far off from the exact growth rates σ_+ of most unstable waves which include very long waves. This leaves very little room for further improvement in the upper bound, especially when we consider the fact that the bound which does not depend on L has to hold for all values of L on which the growth rates depend, though the dependence of exact growth rates on L is exponentially small (see [6]). In [6], it has been shown that the quadratic equation whose solutions are σ_+ and σ_- contains a term involving e^{-kL}, and none of the other terms in the equation depends on L. On the other hand, in Fig. 2(b), 3(b), 4(b), and 5(b), the lower bound shown by dashed curves agrees closely with the actual growth rates σ_- only for modest values of large wavenumbers, and quickly diverges away from the actual growth rates with increasing wavenumber. Thus, there is a lot of room for improving the lower bound result (29). It will be worthwhile in the future to make an attempt at improving upon this lower bound result.

In closing this section, it must be stressed that the bounds (27) and (29) are valid for any values of the parameters, including $L > 0$. As discussed above, the large-k regime for which the bound (29) holds corresponds to all stable modes, which includes modest values of k as well as $k \to \infty$ (see Figs. 2 through 5). In this asymptotic limit, $k \to \infty$ with L finite, $kL \to \infty$, and in this limit, dispersion equation (9) gives either $a = k$ or $b = -k$. Using these values of a and b and the definitions of r, p, s, q from (4) and (5) in (6), it follows that

$$\sigma = [(\mu_r - \mu)Uk - Tk^3]/(\mu_r + \mu) < 0, \quad \text{or} \quad \sigma = [(\mu - \mu_l)Uk - Sk^3]/(\mu + \mu_l) < 0, \tag{30}$$

in the limit $k \to \infty$. This is consistent with the bound σ_{ls} given in (29), keeping in mind that the value of this bound will be negative for large k. The two limiting values of σ given by (30) as $kL \to \infty$ are actually formulas for pure individual Saffman–Taylor growth rates of two individual interfaces. This makes sense, because this limit also includes the limit $L \to \infty$ (for any finite k) when the instabilities of the two interfaces should be independent of each other and should be driven by pure individual Saffman–Taylor instability, i.e., the eigenvalue corresponding to each of the two interfaces should be given by the Saffman–Taylor formula with viscosity jump across that interface only.

4. Conclusions

We have derived here for the first time a lower bound σ_{ls} on the growth rate of short waves and re-derived an upper bound σ_{ul} on the growth rate of long waves. We have also shown that the lower bound is valid for all stable waves, i.e. $\sigma_{\text{ls}} < \sigma^-$ for all $k > \min(k_1, k_2)$, and that the upper bound is valid for all unstable waves, i.e. $\sigma_{\text{ul}} > \sigma^+$ for all $k < \max(k_1, k_2)$. Therefore, we have provided here the upper bound on all the most unstable modes (σ^+) and the lower bound on all the most stable modes (σ^-). Moreover, we have shown that $\sigma_{\text{ls}} = \sigma^- = 0$ at $k = \min(k_1, k_2)$ and $\sigma_{\text{ul}} = \sigma^+ = 0$ at $k = \max(k_1, k_2)$. These have been also validated using numerical results.

These results are useful in many ways. One can use these exact results without resorting to computation to qualitatively estimate the influence of changes of various parameters such as S, T, U, and μ on the growth rates of stable and unstable waves. In [11], stabilization criteria have been given based on an absolute upper bound on the growth rate. However, this does not imply that a stabilized system based on an absolute upper bound (see [11]) will stabilize all individual modal disturbances. The lower and upper bounds (27) and (29) for stable and unstable waves, respectively, can be used to determine the influence of such stabilization on any individual modal disturbance. There are many creative ways one can think of for using these exact results. For example, following the exact procedure outlined in [11], one can find new stabilization criteria (i.e., the values of S, T, μ) based on these local bounds rather than the absolute upper bound, and the purpose of doing so will be to target the stabilization of specific stable or unstable waves.

We should mention in closing the limitations of the upper and lower bounds (27) and (29). These results are certainly valid, for reasons mentioned previously, to predict the onset of instability. Moreover, these bounds contain all the parameters including both interfacial tensions which show that the interaction between the interfaces prevails even within the linearized theory. Thus, there is a transfer of instability between the interfaces regardless of how weak the interfacial disturbances are. As the disturbances grow and the shapes of the interfaces change, nonlinearity comes into play, and these bounds based on linear theory may not hold in the nonlinear regime. Nonetheless, it will be worthwhile to test this using numerical as well as physical experiments, which falls outside the scope of this paper.

Acknowledgments

We would like to thank the referees for making very helpful and constructive suggestions, which have helped us to improve the paper. It is a pleasure to thank Gelu Pasa for his interest in this problem. We thank Xueru Ding for plotting the figures for us. This paper has been made possible by an NPRP grant from the Qatar National Research Fund (a member of The Qatar Foundation). The statements made herein are solely the responsibility of the author.

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