The singularity at the tip of the rising plane bubble: The case of nonzero surface tension

Prabir Daripa
Department of Mathematics, Texas A&M University, College Station, Texas 77843

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In the past pointed bubbles have been obtained numerically in the presence of surface tension. In this paper it is proven that if such pointed bubbles do exist in the presence of surface tension, then the singularity at the corner must be an irregular singular point. The generality and significance of the result are discussed.

In this work, we characterize the singularity at the tip of an unphysical pointed bubble in the presence of surface tension. These bubbles have been recently obtained by Vanden-Broeck[1] in numerical calculations. There are two primary motivations for characterizing this singularity. The first comes from the fact that construction of proper numerical methods in the presence of singularities may require the knowledge of the nature of these singularities. Even then, the numerical issues can be very delicate and may require very careful handling in the construction of numerical methods. The second comes from the fact that it has direct relevance to similar problems of pattern selection in the presence of surface tension (see concluding remarks). In the present context, these physical patterns are round bubbles. The determination of these round bubbles involves solving a nonlinear eigenvalue problem as in Vanden-Broeck[1] with the angle at the apex of the bubble as one of the free parameters. One solves this eigenvalue problem numerically to find the apex angle as a function of the speed of the bubbles for a fixed value of surface tension. The spectrum of round bubbles is found to be discrete and contained in a continuous family of bubbles. This continuous family of bubbles contains sets of a continuous family of pointed bubbles with the round bubbles separating these continuous families of pointed bubbles.[1]

The physical situation here consists of an infinitely long symmetric bubble rising at a constant velocity U in a two-dimensional channel of width h (see Fig. 1).[2] The interior angle at the tip of the bubble is denoted by θ_t. The flow exterior to the bubble is considered inviscid and incompressible. The flow is characterized by its Froude number F and the Weber number W

$$F=U/\sqrt{gh}\,,\quad W=\rho U^2h/T\,. \tag{1}$$

Here g is the gravitational acceleration, T is the surface tension, and ρ is the density of the fluid. The Froude number F refers to the dimensionless speed of the bubble. The principal issue has been the speed of the bubble. This problem is usually solved in a moving reference frame attached to the bubble.

Theoretically this problem has been intractable due to severe nonlinearities in the interface condition. There are no existence or uniqueness theorems for this problem. Birkhoff and Carter[3] were the first to formulate and solve this problem numerically with zero surface tension. They considered only the existence of a unique round bubble and numerically obtained an approximate solution with $F\approx 0.23$. This is consistent with experimental results of Collins[4] with small surface tension. Birkhoff and Carter encountered difficulties with their numerical method on this problem and their numerical results were not very consistent. They attributed these difficulties to the presence of singularities at the tails of the bubble. Garabedian[5] subsequently applied asymptotic methods to this problem and provided analytical evidence that the solution is not unique. He suggested the existence of a continuous family of round bubbles for $F<F_c$ where $F_c\approx 0.23$. Subsequently Vanden-Broeck[6] solved this problem using a numerical method similar to that of Birkhoff and Carter.[3] He obtained the following results: smooth bubbles for $F<F_c$, cusped for $F>F_c$, and pointed with $\theta_t=120°$ for $F=F_c$, where $F_c=0.357\,75$. His results contain the results of Garabedian.[5] In obtaining these results, Vanden-Broeck put special effort in representing the solutions near the tip of the bubble where singularities may appear. For nonzero surface tension, this problem was also solved by Vanden-Broeck[1] numerically. In fact, his computation shows that surface tension makes the problem more singular. He finds that surface tension makes the round zero surface tension bubbles pointed except for a discrete set of velocities. Obviously, the source of these unphysical bubbles in the presence of surface tension is either in the equations or in their discrete analogs used for computations. The purpose here is to explain the origin of this nonphysical behavior from a theoretical standpoint.

At this point it is worthwhile to classify these singularities at the tip. When the problem is formulated in the circle plane, $|\sigma|<1$, an apex angle of θ_t corresponds to an analytic function $\zeta(\sigma)$ having a singularity of order $\gamma=\theta_t/\pi$ at $\sigma=i$ [see Eq. (4) below]. This analytic function $\zeta(\sigma)$ then admits the following representation near this singularity:

$$\zeta(\sigma)=(1+\sigma^2)^\gamma f(\sigma). \tag{2}$$

The singularity is termed regular if $f(\sigma)$ is analytic there, otherwise the singularity is termed irregular. Identification of the nature of these singularities may be useful in devising appropriate numerical methods which can handle such singularities effectively.

In the case of zero surface tension, it can be shown that a corner with 120° interior angle is admissible and that such a corner is probably not a regular singular point. Support for the complicated nature of the singularity at such a corner is based on the analysis of Grant[7] at the crest

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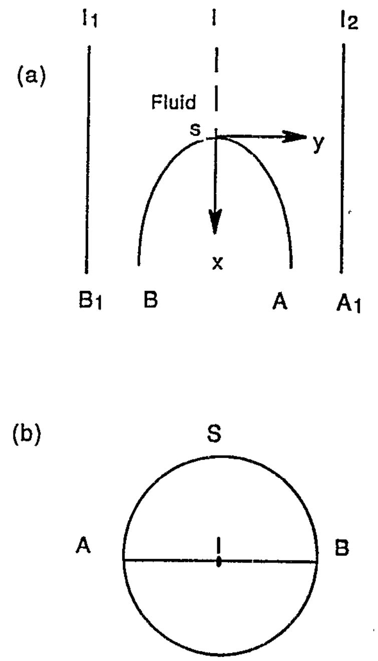

FIG. 1. (a) The physical region: bubble is rising upward in the fluid with speed U. The diameter of the tube is h. (b) The complex σ plane.

of the Stokes wave. *In the case of nonzero surface tension, there is no theoretical proof of the possibility and the nature of such a corner.* In this article we prove that if such pointed bubbles do exist in the presence of surface tension, then the singularity at the corner must be an irregular singular point in the sense described above. Without any loss of generality, nature of the singularity is explored in an auxiliary circle $|\sigma|<1$, since asymptotic and numerical results on this problem have been obtained in this plane.

The plane potential flow past the symmetric bubble can be described in terms of a complex velocity $\zeta=u-iv$. Henceforth we assume that all variables have been made dimensionless by use of (h/U) and h as normalizing constants for time and length dimensions, respectively. Since we have assumed the fluid to be inviscid and incompressible, the complex velocity ζ is an analytic function of the complex potential $w=\phi+i\psi$ in the infinite strip $0<\psi<1$. This strip is the potential plane image of the flow configuration in the tube. Here the slit $\psi=1/2$, $\sigma>0$, is the image of the interface, and the lines $\psi=0$, $\psi=1$ are the walls of the tube. Here ϕ and ψ denote, respectively, the potential and streamfunctions. The conformal mapping

$$e^{-\pi w}=\tfrac{1}{2}(\sigma^{-1}-\sigma) \qquad (3)$$

maps the bubble surface onto the upper semicircle $\sigma=e^{i\alpha}$, $\alpha\in[0,\pi]$, the walls on the real axis, and the flow domain onto the interior of the upper semicircle [see Fig. 1(b)]. The image of the tip of the bubble is $\sigma=i$ (i.e., $\alpha=\pi/2$ on $|\sigma|=1$) and that of the tail of the bubble is $\sigma=\mp 1$. At the tip of the bubble

$$\zeta\approx(1+\sigma^2)^\gamma, \quad \text{as} \quad \sigma\to i, \qquad (4)$$

where $\gamma=\theta_t/\pi$ and $0<\gamma<1$. Let $\beta=\alpha-\pi/2$ so that $\beta=0$ refers to the tip singularity. From the nature of the singularity in Eq. (4), it follows that

$$q=O(\beta^\gamma) \quad \text{and} \quad \theta=-(\gamma\pi/2)+o(\beta), \quad \beta\to 0. \qquad (5)$$

The interface condition is given by the Bernoulli equation

$$q^2-(2/F^2)\,x-[2K(x)/W]=\text{const}, \qquad (6)$$

where K, the curvature, is considered positive when traversing the bubble surface leaves the bubble interior on the left side of the interface. After differentiation, the equation above reduces to

$$qq_\alpha-\frac{1}{F^2}x_\alpha-\frac{(\theta_\alpha\alpha_s)_\alpha}{W}=0, \quad \text{on} \quad \sigma=e^{i\alpha}, \quad 0<\alpha<\frac{\pi}{2}, \qquad (7)$$

where we have used $K=\theta_\alpha\alpha_s$, with s parametrizing the surface of the bubble. It is easy to see that $z_\phi=z_w=q^{-1}e^{i\theta}$ and from Eq. (3), $\phi_\alpha=-\cot\alpha/\pi$ on the unit circle. Therefore it follows that

$$z_\alpha=z_\phi\phi_\alpha=-(\cot\alpha/\pi q)\,e^{i\theta}, \quad 0<\alpha<\pi. \qquad (8)$$

Using Eq. (8) and the variable $\beta=\alpha-\pi/2$ in Eq. (7), we have the following interface condition:

$$q^2q_\beta=\left(\frac{1}{\pi F^2}\right)\tan\beta\cos\theta+\frac{q}{W}(\pi q\theta_\beta\cot\beta)_\beta, \quad 0<\beta<\frac{\pi}{2}. \qquad (9)$$

We will show by *contradiction* that the singularity at the tip of a pointed bubble $(0<\gamma<1)$ in the presence of surface tension is an irregular singular point. Suppose that the tip of the pointed bubble with $0<\gamma<1$ is a regular singular point. Then the following series expansions hold on $|\sigma|=1$ in the neighborhood of the tip $(\beta=0)$

$$q=\beta^\gamma(a_1+a_2\beta+a_3\beta^2+\cdots), \qquad (10)$$

$$\theta=-(\gamma\pi/2)+b_1\beta+b_2\beta^2+\cdots. \qquad (11)$$

Substituting Eqs. (10) and (11) in Eq. (9), one obtains

$$\{\beta^{3\gamma-1}[c_0+c_1\beta+c_2\beta^2+O(\beta^3)]\}$$
$$=\left\{\frac{1}{F^2}\left[\cos\left(\frac{\gamma\pi}{2}\right)d_1\beta+d_2\beta^2+O(\beta^3)\right]\right\}$$
$$+\left\{\frac{\beta^{2\gamma}}{W}[e_1\beta^{-2}+e_2\beta^{-1}+e_3+O(\beta)]\right\}, \qquad (12)$$

where $c_0, c_1,\ldots,d_1, d_2,\ldots,e_1, e_2,\ldots$ are the coefficients. (More precise forms of this equation can be obtained using the complete series expression for the complex velocity $e^{-\tau}$ in terms of σ.)

Some remarks are in order. The origin of the terms within the first curly bracket in Eq. (12) is kinetic energy. Similarly the terms within the second and third curly brackets correspond to potential energy and surface tension, respectively. All of these forces are nonzero away from the stagnation points on the interface. Since the power series in β for the potential energy contains only integer powers of β and all three energy terms must be nonzero, similar terms with integer powers must also appear in the power series originating from kinetic energy

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and surface tension. This implies that γ must be such that both 2γ and 3γ are either zero or positive integers. The allowable values for γ are then 0 and 1 which correspond to a cusped and a round bubble, respectively. This contradicts our assumption that $0<\gamma<1$. Thus we have proven that the singularity at the tip of the pointed bubble must be an irregular singular point.

It should be emphasized that our proof *does not imply* that the tip of the round or cusped bubbles, if these bubbles exist, is a regular singular point. For the case of zero surface tension ($W=\infty$), we obtain the familiar result: $\gamma=1$, 2/3, and 0, even though this *does not imply* that the tip is a regular singular point for these values of γ. Although the nature of the singularities for these cases is not known, it should be pointed out that by matching coefficients of like orders in β in Eq. (12), infinitely many conditions relating local properties of the flow variables at these singularities can be generated, and these conditions should hold if these singularities were regular. They will serve at least as necessary conditions for the singularity to be regular and may provide useful if these singularities were irregular. However, in practice, one truncates the system and thus has only a finite number of these conditions. Therefore, at best there can be strong computational evidence that a singular point is regular.

At this point we may restate our main result by saying that *a necessary condition for nonphysical (pointed) bubbles to be allowable solutions of the governing equations is that the tip must be an irregular singular point [i.e., the function $f(\sigma)$ in Eq. (2) is not analytic].* This condition, together with numerically generated pointed bubbles by Vanden-Broeck,[1] provides strong evidence that the tip is an irregular singular point. In closing, we comment on the generality of our result and on how it sheds light on this problem.

It is useful to note that the origin of the irregular nature of the singularity can be traced in the presence of two distinct types of terms in the nonlinear boundary condition (9): one type containing terms with θ only (no q) and the other type containing polynomials in q and/or θ, with the qualification that $\ln q$ and θ are complex conjugate to each other. The assumption of regularity of the corner singularity in the sense of Eq. (2) then generates two different types of series from the above two terms: one containing integer powers and other containing noninteger powers in a suitable variable. Equating like power terms from these series then does not allow the corner singularity (i.e., noninteger values of γ here), suggesting that the behavior of the corner singularity, if it exists, is not compatible with the form (2).

It is worth mentioning the generality of our result by pointing out the similarity of this problem with the related Saffman–Taylor problem. Similar to the present problem, the pointed fingers have also been obtained in the presence of surface tension when the boundary condition about the included angle at the apex of the finger is relaxed from the equations being solved.[8] (We do not discuss the Saffman–Taylor problem in detail here. Some specifics of this related problem which are relevant to the present issue can be found in McLean and Saffman[9] and also in Vanden-Broeck.[8]) The nonlinearity in the boundary condition [see Eq. (19) in McLean and Saffman[9] or Eq. (2) in Vanden-Broeck[8]] for the Saffman–Taylor problem is weaker than the same [Eq. (9)] for the bubble problem due to the simplicity of the pressure-velocity relation (equivalent to the Darcy's Law in porous media) for the Saffman–Taylor problem. However, these boundary conditions are similar in the types of terms they contain: terms with θ only (no q) and the others containing q and/or θ. Since this pattern is the main source of our result discussed above, it follows that a similar formulation and analysis of the Saffman–Taylor problem would reveal that the corner singularity of the pointed fingers in the case of nonzero surface tension is also irregular.

Finally, we should explain as to why our approach sheds some insight on this problem. To explain this, it is useful to restate precisely what the problem was. Essentially it was the numerical evidence of nonphysical (pointed) bubbles[1] in the presence of surface tension which were, otherwise, absent in the case of zero surface tension (Vanden-Broeck[6]). This seemingly odd behavior may seem to suggest at first that the nonphysical bubbles are possibly spurious, i.e., not allowable solutions of the equations. However, this would be the case if $f(\sigma)$ in Eq. (2) were analytic as suggested by the main result of our analysis. Therefore the nonphysical behavior described above can be resolved by the nonanalyticity of $f(\sigma)$ in Eq. (2), i.e., by the irregular nature of the singularity at the pointed tip. Singularities of a similar nature have been obtained in the past at the crest of a finite amplitude Stokes wave (Grant[7]). In conclusion, we may mention that there are similar problems such as cavitating flows (Vanden-Broeck[10]) where singularities (infinite curvature) are known to appear in the presence of surface tension.

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