

## NOTE ON FINITE DIFFERENCE APPROXIMATIONS TO BURGERS' EQUATION\*

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**Abstract.** Standard finite difference approximations to Burgers' equation are considered from the point of view of dynamical systems theory. Phase plane analyses for discretizations with a few grid points are presented. These show the existence of initial conditions leading to spurious solutions with unlimited amplitude growth due to nonconservation of kinetic energy by the nondissipative terms in the discretizations. It is shown that such solutions may be found even for arbitrarily fine pointwise resolution, i.e., for arbitrarily many grid points. On the other hand, an energy conserving discretization of the nondissipative terms removes all spurious solutions of this kind. The results obtained seem to complement recent investigations of the steady state problem.

**Key words.** finite difference scheme, Burgers' equation, phase plane analysis

**1. Introduction.** We have been studying standard finite difference approximations to Burgers' equation [1] as part of an attempt to compare various numerical methods for solving partial differential equations. Burgers' equation is a valuable test case for such studies since it is directly solvable by the Cole-Hopf [2], [3] transformation, and numerically it is accessible by standard finite difference, finite element and spectral methods. One may also formulate a "particle method" for Burgers' equation by appealing to the possibility of a pole decomposition [4] for this equation.

Our primary interest is in the dynamics of two-dimensional flow, particularly the two-dimensional Euler equation and the representation of its solutions by an assembly of point vortices [5]–[7]. The pole decomposition of Burgers' equation can be seen as an analogue of the vortex decomposition of two-dimensional incompressible hydrodynamics formulated as a field theory for the stream function [7]. Hence, a comparison of "standard" numerical techniques, such as finite differences, for Burgers' equation with the pole decomposition solutions suggests itself. As a preliminary to this a study of the finite difference equations themselves was performed, and, since the ideas of pole and vortex decomposition quickly lead to notions from dynamical systems theory, the finite difference equations were considered from this point of view. There has recently been much interest in using results from the theory of dynamical systems to study in greater detail the nature of the instabilities to which numerical schemes are susceptible [8], [9].

A standard finite difference approximation to Burgers' equation consists of a set of ordinary differential equations, one for the field value at each grid point, coupled through quadratic interactions. As is well known, problems of precisely this format may display chaotic solutions [10]. The Lorenz equations [11] are a case in point. If such behavior occurs for a finite difference approximation to Burgers' equation, it must clearly come from the numerical scheme, since the continuum Burgers' equation is in some sense "integrable." (Burgers' equation is dissipative, and so integrability is not immediately defined. However, its pole decomposition equations can be imbedded in an integrable Hamiltonian system, the Calogero-Moser system [4]. Taffin [12] discusses the concept of integrability and Burgers' equation.)

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\* Received by the editors August 5, 1982, and in revised form April 29, 1983. This research was supported by the National Science Foundation under grant MEA 81-16910 to Brown University.

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We must state right away that we did not find chaotic behavior for our standard finite difference approximations to Burgers' equation. However, the dynamical systems point of view suggested that we look in detail at the "phase plane" for discretizations with a small number of grid points. This was done for  $N = 3, 4, 5$  and  $6$ , and we found for  $N = 4$  that there exist initial states which grow in time beyond all bounds. Such solutions are physically unacceptable. Moreover, we are able to show, by a rather obvious argument, that for any  $N$  which is a multiple of  $4$  such initial states with spurious long time behavior will exist. This result seems to complement recent work on the *steady* Burgers' equation [13], [14]. The result is unsettling because it shows that even for arbitrarily fine pointwise spatial resolution the standard finite difference approximation to Burgers' equation can give physically unacceptable results for certain initial conditions. A complete resolution of this "paradox" is not given. However, we do show that if a slightly different finite difference approximation is employed, which conserves the discretized kinetic energy, then no initial conditions can lead to the above pathology of infinite amplitude growth.

We must emphasize that although our entire discussion centers on Burgers' equation our objective is not to solve that equation numerically. Burgers' equation is trivial. However, as just mentioned, *any* finite difference approximation to the material derivative of a field results in a system of ODEs with quadratic couplings. Thus, we submit that the method of analysis exemplified here is of general applicability and usefulness. (For a related discussion, involving the Fourier amplitude equations for two-dimensional flow, see [15]).

**2. Preliminaries.** We are concerned with Burgers' equation,

$$u_t + uu_x = \nu u_{xx},$$

for a real field,  $u = u(x, t)$ , and specifically consider the initial value problem:

$$u(x, 0) = u_0(x)$$

for periodic boundary conditions on an interval of length  $L$ :

$$u(x + L, t) = u(x, t).$$

To solve this problem numerically, we introduce the grid values  $u_k(t) = u(kL/N, t)$  for  $k = 0, \dots, N-1$ , and discretize  $u_{xx}$  and  $uu_x$  according to

$$u_{xx} = \left(\frac{N}{L}\right)^2 (u_{k+1} + u_{k-1} - 2u_k),$$

$$uu_x = \left(\frac{N}{2L}\right) u_k (u_{k+1} - u_{k-1}).$$

These expressions are accurate to  $O((\Delta x)^2)$ , where  $\Delta x = L/N$ . Substituting into Burgers' equation we obtain a system of coupled ODEs of first order for the amplitudes  $u_k$ . We may nondimensionalize these equations by setting

$$v_k(s) = \left(\frac{L}{N\nu}\right) u_k \left(\frac{L^2 s}{N^2 \nu}\right), \quad k = 0, \dots, N-1.$$

The proposed discretization of Burgers' equation then reads

$$(1) \quad \frac{dv_k}{ds} = -\left(\frac{v_k}{2}\right) (v_{k+1} - v_{k-1}) + v_{k+1} + v_{k-1} - 2v_k.$$

There is one ODE for each grid point amplitude  $v_k$ ,  $k = 0, \dots, N-1$ , making a system of  $N$  coupled equations in all. As noted previously, the couplings are quadratic. The periodic boundary condition now means that  $v_{k+N} = v_k$  for all nondimensional times  $s$ .

The discretization (1) conserves momentum in the sense that

$$\sum_{k=0}^{N-1} \frac{dv_k}{ds} = 0.$$

Since the value of the momentum of the field may be altered at will by subjecting it to a Galilean transformation, viz.

$$U(x, t) = u(x - ct, t) + c,$$

we shall consistently assume that the total momentum vanishes. For the discrete system we thus assume:

$$\sum_{k=0}^{N-1} v_k = 0.$$

This restriction on the sum of the  $v_k$  will prove very convenient later.

Kinetic energy, on the other hand, is not conserved by the scheme (1), i.e.

$$\sum_{k=0}^{N-1} v_k \frac{dv_k}{ds} \neq 0,$$

even if the dissipative (linear) terms are omitted. It may be shown that the kinetic energy of the field  $u$ ,

$$E_{\text{kin}} \equiv \frac{1}{2} \int_0^L u^2 dx,$$

satisfies the equation of motion

$$\frac{dE_{\text{kin}}}{dt} = -\nu \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx$$

and thus decreases unless  $u$  is constant in  $x$  or  $\nu = 0$ . Hence solutions to the discretized equations that make the energy increase indefinitely or solutions that are steady in time but vary in  $x$  are not physically acceptable and must be classified as artifacts of the numerical scheme. We shall meet with such solutions in the next section. In § 4 we shall then trace the origin of these spurious solutions to the fact that scheme (1) does not conserve kinetic energy in the nondissipative limit.

**3. Case studies for small  $N$ .** For  $N = 2$  we are led to consider the system

$$\dot{v}_0 = -\left(\frac{v_0}{2}\right)(v_1 - v_1) + v_1 + v_1 - 2v_0 = 2(v_1 - v_0),$$

$$\dot{v}_1 = 2(v_0 - v_1)$$

(where the dot signifies a derivative with respect to  $s$ ) so that

$$\dot{v}_0 - \dot{v}_1 = -4(v_0 - v_1).$$

Thus  $v_0 - v_1$  decays exponentially and since  $v_0$  and  $v_1$  sum to zero (total momentum is assumed to vanish; see § 2) we see that both  $v_0$  and  $v_1$  must decay to zero. This is completely in accord with our expectations for the continuum equation.

For  $N = 3$  we get the more interesting system

$$\dot{v}_0 = -\left(\frac{v_0}{2}\right)(v_1 - v_2) + v_1 + v_2 - 2v_0,$$

$$\dot{v}_1 = -\left(\frac{v_1}{2}\right)(v_2 - v_0) + v_2 + v_0 - 2v_1,$$

$$\dot{v}_2 = -\left(\frac{v_2}{2}\right)(v_0 - v_1) + v_0 + v_1 - 2v_2.$$

It is not difficult to see that this system has the integral

$$I \equiv v_0 v_1 v_2 \exp(9s)$$

(when  $v_0 + v_1 + v_2 = 0$ ). This integral immediately shows us that if the equations for  $N = 3$  have a steady state solution, one of the amplitudes, say  $v_0$ , must vanish. But if  $v_0 = 0$ , we have  $v_1 = -v_2 \equiv v$  and the system reduces to

$$\dot{v} = \left(\frac{v}{2}\right)(v - 6)$$

with steady states corresponding to  $v = 0$  and  $v = 6$  and in general the solution

$$v(s) = \frac{6v(0) \exp(-3s)}{6 - v(0)(1 - \exp(-3s))}.$$

Figure 1 shows projections of several phase space trajectories, which reside in the plane  $v_0 + v_1 + v_2 = 0$ , onto the  $(v_1, v_2)$ -plane. We see that for certain initial conditions

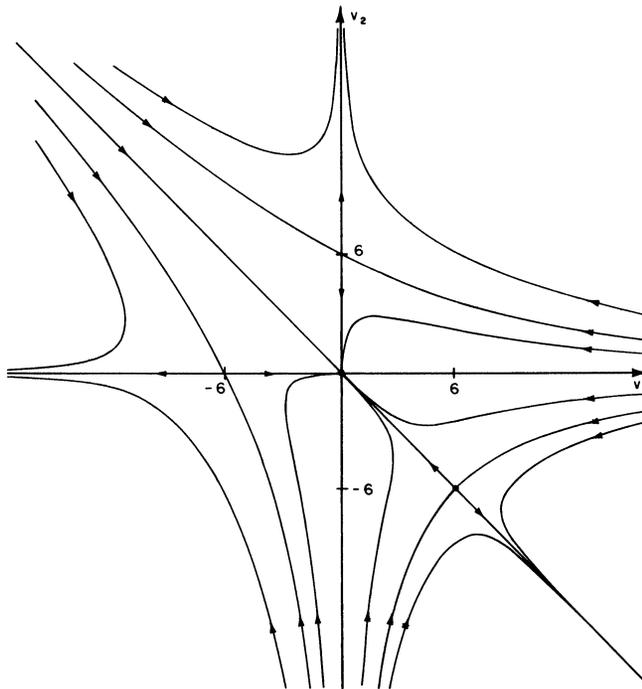


FIG. 1. Phase trajectories, projected onto the  $(v_1, v_2)$ -plane for scheme (1) with  $N = 3$ .

the trajectories depart to infinity due to the existence of saddle points at  $(0, 6, -6)$ ,  $(-6, 0, 6)$  and  $(6, -6, 0)$ .

For  $N = 4$  we must consider the system

$$\begin{aligned}\dot{v}_0 &= -\left(\frac{v_0}{2}\right)(v_1 - v_3) + v_1 + v_3 - 2v_0, \\ \dot{v}_1 &= -\left(\frac{v_1}{2}\right)(v_2 - v_0) + v_2 + v_0 - 2v_1, \\ \dot{v}_2 &= -\left(\frac{v_2}{2}\right)(v_3 - v_1) + v_3 + v_1 - 2v_2, \\ \dot{v}_3 &= -\left(\frac{v_3}{2}\right)(v_0 - v_2) + v_0 + v_2 - 2v_3.\end{aligned}$$

We notice that

$$\begin{aligned}v_3 \dot{v}_1 + \dot{v}_3 v_1 &= (v_1 + v_3)(v_2 + v_0) - 4v_1 v_3, \\ v_0 \dot{v}_2 + \dot{v}_0 v_2 &= (v_0 + v_2)(v_1 + v_3) - 4v_0 v_2.\end{aligned}$$

Hence

$$\frac{d}{ds}(v_0 v_2 - v_1 v_3) = -4(v_0 v_2 - v_1 v_3)$$

and the system has the integral

$$J \equiv (v_0 v_2 - v_1 v_3) \exp(4s).$$

We now observe that the full four-dimensional system has a discrete symmetry: The constraint  $v_0 = -v_3 \equiv U$ ,  $v_1 = -v_2 \equiv V$  will be preserved by the equations of motion. The evolution of  $U, V$  is governed by

$$\begin{aligned}\dot{U} &= -\left(\frac{U}{2}\right)(U + V) + V - 3U, \\ \dot{V} &= \left(\frac{V}{2}\right)(U + V) + U - 3V.\end{aligned}$$

Let

$$X = (U + V)\sqrt{2}, \quad Y = U - V.$$

Then the equations for  $U$  and  $V$  may be written as

$$\dot{X} = \frac{\partial G}{\partial X}, \quad \dot{Y} = \frac{\partial G}{\partial Y}$$

with

$$G(X, Y) = -X^2\left(1 + \frac{1}{4}Y\right) - 2Y^2.$$

Several level curves of the potential  $G$  are shown in Fig. 2a and the  $(X, Y)$ -flow, which arises as the family of trajectories orthogonal to the level curves, is shown in Fig. 2b. Again we see spurious steady states and again they apparently give rise to modes of evolution in which some discrete amplitudes grow indefinitely.

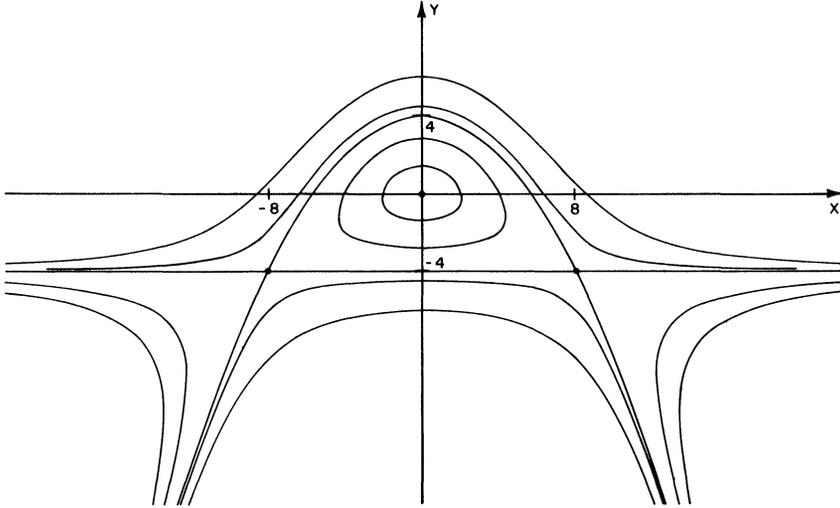


FIG. 2(a). The potential  $G(X, Y)$  pertaining to the analysis of scheme (1) with  $N = 4$ . The separatrix consists of the line  $Y = -4$  and the parabola  $Y = -\frac{1}{8}X^2 + 4$ .

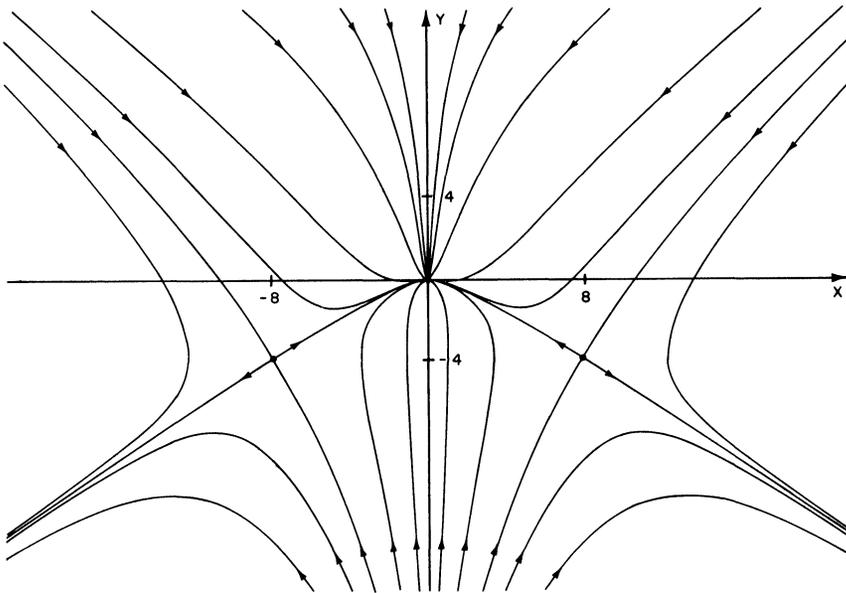


FIG. 2(b). The  $(X, Y)$ -flow pertaining to scheme (1) with  $N = 4$ .

We conclude this section by observing that increasing  $N$  will not rule out the existence of initial conditions leading to physically spurious solutions. This follows, for example, if  $N = 4n$ . The finite difference amplitude equations will then have a discrete symmetry which consists in every fourth amplitude being the same, i.e.,  $v_{k+4p} = v_k$  where  $k = 0, 1, 2, 3$  and  $p = 0, 1, \dots, n-1$ . Within the subspace singled out by this symmetry, the system of ODEs reduces to  $n$  replicas of the  $N = 4$  system discussed above. Now consider an initial condition with the repeated period-four symmetry in

the region that leads to indefinite growth. Such an initial condition can be found in the form

$$(v_0, \dots, v_{N-1}) = (a, b, -a, -b, a, b, -a, -b, \dots, a, b, -a, -b)$$

according to our analysis of the  $N=4$  system above. Since the region in the  $(v_0, v_1, -v_0, -v_1)$  subspace that leads to indefinite growth is obviously an open set (cf. Fig. 2b), it must be possible to find initial conditions for the full  $N=4n$  system of the form

$$(v_0, \dots, v_{N-1}) = (a + e_1, b + e_2, -a + e_3, -b + e_4, \dots)$$

where  $e_1, e_2, e_3, e_4$ , etc. are different, i.e. initial conditions *without* the discrete, period four symmetry, that still lead to indefinitely large amplitudes. We have tried several such initial conditions and checked numerically that they indeed lead to unlimited amplitude growth.

To see what this property of scheme (1) means in terms of the original variables  $u_k$  we must refer to the definition of  $v_k$  in § 2. Then we see that if  $u_k(0) = F_k$ ,  $k = 0, 1, 2, 3$ , is a set of initial amplitudes that leads to indefinite growth for a discretization with 4 grid points,  $u_{k+4p}(0) = nF_k$ ,  $k = 0, 1, 2, 3$ ,  $p = 0, 1, \dots, n-1$ , will lead to indefinite growth for a discretization with  $N = 4n$  grid points. We shall restate this result in terms of the so-called *cell Reynolds number* in § 5.

**4. Remedy.** Having described the pathologies of the scheme (1), we must now prescribe a cure. We note that all the spurious solutions in § 3 violate the requirement that kinetic energy be dissipated. Indeed with diverging amplitudes the discretized kinetic energy clearly tends to infinity. Hence, if a scheme that conserves energy (when the dissipative terms are neglected) can be found, unbounded spurious solutions, such as those found for scheme (1), should disappear. It is not difficult to find such a scheme: We first recall that Burgers' equation may be written in "conservation form"

$$u_t + (\frac{1}{2}u^2)_x = \nu u_{xx},$$

and this form can then be discretized. This leads to

$$(2) \quad \frac{dv_k}{ds} = -\frac{1}{4}(v_{k+1}^2 - v_{k-1}^2) + v_{k+1} + v_{k-1} - 2v_k$$

with the same rescalings as before. This scheme again conserves momentum but not energy. Few-amplitude truncations of the system (2) can be shown to display spurious solutions as in § 3.

However, we now have two schemes and new schemes can be constructed by forming convex combinations of them. In particular, we can ask whether some such combination will conserve energy. Thus we add (1) and (2) with "weights"  $w$  and  $1-w$  respectively and impose the condition that the combination conserve energy (when the linear, dissipative terms are neglected). This turns out to determine  $w$  uniquely ( $w = \frac{1}{3}$ ) and the resulting scheme is [16]

$$(3) \quad \frac{dv_k}{ds} = -\frac{1}{6}(v_{k+1} - v_{k-1})(v_{k+1} + v_{k-1} + v_k) + v_{k+1} + v_{k-1} - 2v_k.$$

Since it arose by linear (convex) combination of (1) and (2), scheme (3) clearly still conserves momentum. Using the Cauchy-Schwarz inequality it is also easy to show

from (3) that

$$\sum_{k=0}^{N-1} v_k \frac{dv_k}{ds} \leq 0$$

where the equal sign only holds if all  $v_k$  are identical (and hence zero).

We remark that it may sometimes be undesirable to have an energy-conserving discretization (in the limit of zero viscosity) for physical reasons. For example, if the objective is to track shock formation in an initial value calculation, energy conservation may actually be an unwanted constraint [17, p. 252].

**5. Concluding remarks.** It is useful to state precisely what the remedy of § 4 was. Essentially it consisted in discretizing the factor  $u$  of  $uu_x$  by  $(v_{k+1} + v_{k-1} + v_k)/3$  instead of just by  $v_k$  (compare (3) to (1)). This kind of differencing to produce the value of the field itself at a point, as opposed to the values of derivatives, arises frequently and naturally in the finite element method. In fact, the discretization of the convective derivative in (3) can be obtained using the finite element method with a basis of piecewise linear functions.

It is worth reiterating that we have not found evidence of chaotic behavior. This seems to be due to the absence of any free parameters in our discretized equations. With the relative magnitude of linear and nonlinear terms that is forced upon us here, the phase space flows seem to be dominated by sinks and saddles. We have not attempted to insert a variable parameter into these equations in order to seek out regimes of chaotic solutions since the physical significance of such an exercise seemed unclear.

We may restate our results by saying that close to the origin even the "naive" schemes (1) and (2) give qualitatively acceptable results. In fact, we can identify a certain region,  $|v_k| \leq v_{\max}(N)$  for  $k = 0, 1, \dots, N-1$ , within which the discretization behaves in a qualitatively correct way compared to the continuum equation. In terms of the field amplitudes  $u_k$  and the spatial resolution  $\Delta x$  (§ 2) this criterion takes the form  $|u_k| \Delta x / \nu \leq v_{\max}(N)$ , i.e. the cell Reynolds number,  $\text{Re} \equiv (\max_k |u_k|) \Delta x / \nu$ , must be chosen less than some  $N$ -dependent upper bound. Clearly if  $v_{\max}(N)$  tends to infinity with increasing  $N$ , spurious solutions become less troublesome with increasing resolution. However, we have just seen that for scheme (1),  $v_{\max}(4n) \leq v_{\max}(4)$  (see the argument given at the end of § 3) and thus that  $v_{\max}(N)$  does not increase systematically in this case. In terms of the original variables  $u_k$  we must go to ever larger amplitudes as  $N$  increases to encounter the spurious behavior. But for scheme (1) spurious solutions can be found; for scheme (3), they cannot. We may mention in conclusion that for cell Reynolds number less than 2 scheme (1) is usually found to be adequate [13], [14].

**Acknowledgments.** We are indebted to Professor T. F. Morse for comments and discussion. One of us (P.K.D.) would like to thank the Graduate School at Brown University for fellowship support.

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