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Trapped modes in a channel containing three layers of fluids and a submerged cylinder

A. Chakrabarti, P. Daripa* and Hamsapriye

Abstract. The problem of existence of trapped waves in fluids due to a cylinder is investigated for the hydrodynamic set-up which involves a horizontal channel of infinite length and depth and of finite width containing three layers of incompressible fluids of different constant densities. The set-up also contains a cylinder which is impermeable, fully immersed in the bottom (lower-most) fluid layer of infinite depth, and extends across the channel with its generators perpendicular to the side walls of the channel. When the ratios of the densities of the adjacent fluids differ from unity by sufficiently small quantities, the underlying mathematical problem reduces to a generalized nonlinear eigenvalue problem involving a cubic polynomial-cum-operator equation. The perturbation analysis of this eigenvalue problem suggests existence of three distinct modes with different frequencies: one of the order of one persisting at the free surface, and the other two of the order of the density ratio (except for modulo one) persisting at the two internal interfaces. The correlation between these results for the three-layer case and very recent numerical results of other authors in the two-layer case has also been addressed.

Keywords. Trapped waves, potential flows, interfacial flows.

1. Introduction

There have been propositions in many scientific circles for the development of underwater tubular bridges across several fjords and straits (see [1]) as a viable means of improving/increasing modes of transportation. However, undertaking of such projects requires, among many other concerns, investigation of various hydrodynamic phenomena that may arise due to the presence of such tubular bridges. Some of these phenomena may need to be taken into consideration for the design and construction of such bridges. One such phenomenon is the occurance of trapped waves: time harmonic oscillations of fluid particles with its amplitude decaying to zero in the far field. From the 1951 work of Ursell [2], such trapped waves with one set of frequencies have been known to appear due to the presence of a cylinder extending across a channel in an incompressible fluid layer of constant density. Such studies on trapped waves, due to a cylinder in an incompressible fluid

*Author for correspondence.

of infinite depth, by several researchers (see Ursell [2], Jones [3]) have led to the development of interesting mathematical techniques to solve a class of boundary value problems involving an unknown parameter ν associated with the frequency of such trapped waves.

Since straits and fjords rarely contain a fluid of constant density, it seems that a more appropriate model for studies of existence and properties of such waves should perhaps include more than one fluid layer because the number and frequencies of such trapped waves, if they exist in multi-layer cases, may depend on the number of such fluid layers and their densities. In fact, recently Kuznetsov [4] has investigated such trapped waves in a channel containing two layers of incompressible fluids of different constant densities and a fully immersed impermeable cylinder extending across the channel in the bottom layer. This model is based on the assumption that the top layer is perhaps pure water and the bottom layer is salty with density only slightly more than that of pure water (we comment more on this model later in this section). Kuznetsov's theoretical analysis of the problem gives rise to a quadratic-cum-operator nonlinear eigenvalue problem from which he concludes that, in contrast to the one-layer case, there are two sets of frequencies at which trapped modes might exist in this two-layer case: one set (having frequency of the order one) corresponding to disturbances on the free surface which are similar to those for the one-layer fluid case, and the other set (having frequency of the order of the density difference) corresponding to trapped modes on the interface between the two layers,

The sharp interface between the two layers in the above two-layer model is basically a crude representation of the smooth pycnocline that exists between fresh and salt water. However, a better model perhaps would involve replacing the sharp interface in the two-layer model with a layer of finite width in which the density either varies linearly between upper and lower values, or remains constant representing some sort of mean density of the middle layer. Even though the first case is perhaps a better model, the later case is simpler and serves to improve upon the two-layer model. In this paper, we study the simpler three-layer fluid problem, each layer having constant density, using Kuznetsov's theoretical approach. In particular, we address the problem of existence of trapped waves in a channel containing three layers, two of which are of finite depths, and a submerged cylinder in the bottom layer of infinite depth. This problem reduces to the above two problems (one-layer problem of Ursell [2] and two-layer problem of Kuznetsov [4]) in the limiting cases (see next section).

Using mathematical tools similar to the ones used in [4], we find that the answer to our problem of existence of trapped waves in the three layer case lies in addressing the issue of existence of a parameter ν associated with the frequency of trapped modes. It is shown here that this parameter ν satisfies a cubic polynomial-cum-operator equation which generalizes the corresponding quadratic equation that arose in the study of two-layer set-up by Kuznetsov [4].

The generalized problem considered here contains two small parameters ϵ and

ϵ^*, involving the ratios of the densities of the adjacent layers of fluids under consideration (see Sec. 2 below), unlike in the work of Kuznetsov [4] where there is only a single parameter ϵ. When $\epsilon^* = \alpha\epsilon$ with α a fixed known constant, we show, using a perturbation procedure similar to that employed in [4], that there exists three sets of frequencies (positive eigenvalues) ν which correspond to three distinct trapped modes of waves. For the hydrodynamic set-up considered here, the first set of frequencies corresponds to trapped modes of waves on the free surface, and the other two sets correspond to trapped modes of waves on the two internal interfaces. This is in contrast with the two sets of trapped modes obtained by Kuznetsov [4] in the case of two layers, the first set of which corresponds to the trapped modes of waves on the free surface and the second set corresponds to the trapped modes of waves on the internal interface.

It is worth emphasizing here that the limiting case $\epsilon \to 0\,(\alpha > 0)$ of our problem (see text) corresponds to a one-layer fluid of infinite depth, the one considered by Ursell [2]. We show below that our results derived in this paper for three-layer case recover Ursell's results in this limit. The $\alpha \to 0\,(\epsilon > 0)$, limit problem corresponds to the two-layer fluid problem and our results in this limiting case are consistent with the results of Kuznetsov [4]. Some of these results are discussed in light of very recent numerical results of Linton and Cadby [5] on the trapped modes in the two-layer problem which is a limiting case of our three-layer problem.

The layout of the paper is as follows. The problem is defined and formulated in Sec. 2. In Sec. 3, the problem is reduced to that of determining a single unknown potential which describes motion of the fluid in the bottom layer. In Sec. 4, with the aid of the special Green's function constructed by Ursell [2], the problem is reduced further to a cubic eigenvalue problem involving an eigenvalue and a set of known operators. The eigenvalue problem derived in section 4 is analyzed in Sec. 5 using a perturbation approach and the results obtained are examined for the limiting values of the parameters. Finally we conclude in Sec. 6.

2. Statement of the problem

We consider three layers of incompressible fluid in a channel of finite width but of infinite length and depth with vertical side walls. The top layer W^{**} contains fresh water of density ρ^{**} and has depth $(1-\delta)$ where $0 < \delta < 1$. The middle layer W^* has salt water of constant density ρ^* and has depth $\delta > 0$. The bottom (lower-most) layer W contains muddy water of another constant density ρ and is of infinite depth. The problem is formulated in a Cartesian (x, y, z) coordinate system with y-axis directed normal to the bottom internal interface separating the middle and the bottom layers, and the (x, z)-plane coinciding with this internal interface. The side walls are at $z = \mp b$. The bottom layer has a fully immersed cylinder extending across the channel between its side walls and it has its generators parallel to the z-axis, i.e. perpendicular to the side walls. Below, the arbitrary cross section of the cylinder is denoted by D and its rigid boundary by S. The mathematical formulation of the problem under consideration is described below assuming linearized theory of surface water waves.

We define $\phi(x,y,z,t), \phi^*(x,y,z,t)$, and $\phi^{**}(x,y,z,t)$ as the time dependent velocity potentials corresponding to the irrotational motion of the muddy, salty and fresh water respectively. It is easy to see that these potentials satisfy the following equations and conditions (see also [4]).

$$\nabla^2\phi^{**} \equiv \phi^{**}_{xx} + \phi^{**}_{yy} + \phi^{**}_{zz} = 0, \text{ in } W^{**}, \tag{2.1}$$

$$\phi^{**}_{tt} + g\phi^{**}_y = 0, \text{ on } y = 1, \tag{2.2}$$

$$\phi^{**}_z = 0, \text{ on } z = \pm b, \tag{2.3}$$

$$\nabla^2\phi^* = 0, \text{ in } W^*, \tag{2.4}$$

$$\phi^{**}_y = \phi^*_y, \text{ on } y = \delta, \tag{2.5}$$

$$\rho^{**}(\phi^{**}_{tt} + g\phi^{**}_y) = \rho^*(\phi^*_{tt} + g\phi^*_y), \text{ on } y = \delta, \tag{2.6}$$

$$\phi^*_z = 0, \text{ on } z = \pm b, \tag{2.7}$$

$$\nabla^2\phi = 0, \text{ in } W, \tag{2.8}$$

$$\phi^*_y = \phi_y, \text{ on } y = 0, \tag{2.9}$$

$$\rho^*(\phi^*_{tt} + g\phi^*_y) = \rho(\phi_{tt} + g\phi_y), \text{ on } y = 0, \tag{2.10}$$

$$\phi_z = 0, \text{ on } z = \pm b, \tag{2.11}$$

$$\phi_n \equiv \frac{\partial\phi}{\partial n} = 0, \text{on } S, \tag{2.12}$$

and

$$\phi \to 0, \text{as } y \to -\infty. \tag{2.13}$$

In (2.12) above, S refers to the boundary of the cross-section of the cylinder. The above relations are the usual ones in the theory of linear hydrodynamics, which arise from the conditions of continuity of the normal component of the velocity field and the pressure across the interfaces $y = 0$, and $y = \delta$. The surface tension at the interfaces and the free surface is neglected. We assume that

$$\rho = \rho^*(1+\epsilon), \text{ and } \rho^* = \rho^{**}(1+\epsilon^*) \text{ for } \epsilon, \epsilon^* > 0, \tag{2.14}$$

where the positive quantities ϵ and ϵ^* are assumed to be sufficiently small for the analysis, described in section 3, to hold good.

For the problem defined by (2.1) through (2.14), we look for trapped wave solutions of the form:

$$\begin{aligned} \phi(x,y,z,t) &= e^{-i\omega t}u(x,y)\cos(kz), \\ \phi^*(x,y,z,t) &= e^{-i\omega t}u^*(x,y)\cos(kz), \\ \phi^{**}(x,y,z,t) &= e^{-i\omega t}u^{**}(x,y)\cos(kz), \end{aligned} \tag{2.15}$$

where ω corresponds to the frequency of the waves and k is the wave number.

In order that the boundary conditions on the side walls (i.e. the conditions (2.3), (2.7), and (2.11)) are satisfied, we set

$$k = \frac{\pi n}{b}, \; n = 1, 2, \cdots \tag{2.16}$$

Using the relations (2.15) in the relation (2.1)-(2.13), and defining

$$\nu = \frac{\omega^2}{g}, \tag{2.17}$$

we obtain that

$$\nabla^2 u^{**} = k^2 u^{**}, \text{ in } W^{**}, \tag{2.18}$$

$$u_y^{**} - \nu u^{**} = 0, \text{ on } y = 1, \tag{2.19}$$

$$u_y^{**} = u_y^{*}, \text{on } y = \delta, \tag{2.20}$$

$$\nabla^2 u^{*} = k^2 u^{*}, \text{ in } W^{*}, \tag{2.21}$$

$$\rho^{**}(u_y^{**} - \nu u^{**}) = \rho^{*}(u_y^{*} - \nu u^{*}), \text{on} y = \delta, \tag{2.22}$$

$$u_y^{*} = u_y, \text{on } y = 0, \tag{2.23}$$

$$\rho^{*}(u_y^{*} - \nu u^{*}) = \rho(u_y - \nu u), \text{ on } y = 0, \tag{2.24}$$

$$\nabla^2 u = k^2 u, \text{ in } W, \tag{2.25}$$

$$u_n \equiv \frac{\partial u}{\partial n} = 0, \text{ on } S, \tag{2.26}$$

$$u(x, y) \to 0, \text{ as } y \to -\infty, \tag{2.27}$$

For trapped mode solutions, we also need that the motion decay at large distances, i.e.,

$$\left.\begin{array}{lll} u^{**}, |\nabla^2 u^{**}| \to 0, & \text{as} & |x + iy| \to \infty, \\ u^{*}, |\nabla^2 u^{*}| \to 0, & \text{as} & |x + iy| \to \infty, \\ u, |\nabla^2 u| \to 0, & \text{as} & |x + iy| \to \infty. \end{array}\right\} \tag{2.28}$$

In the next two sections, through some manipulations we reduce the problem defined by (2.18) through (2.28) involving the three unknown functions u, u^*, u^{**}, to an equivalent but simpler problem which can be analyzed by the tools of the spectral theory of operators.

3. Reduction to the problem in the lower-most layer

Using the Fourier transform as defined by the relation

$$\tilde{u}(\xi, y) = \int_{-\infty}^{\infty} u(x, y) e^{-ix\xi} \, dx, \tag{3.1}$$

all the equations and conditions (2.18)-(2.25) are transformed easily and the resulting ordinary differential equations for the transformed unknowns $\tilde{u}^{**}$ and $\tilde{u}^{*}$ are solved in terms of the function $\tilde{u}_y(\xi,0)$ along the lines described by Kuznetsov [4]. In doing so, we find that the condition (2.23) can be expressed as

$$\tilde{u}_y(\xi,0)=\frac{\nu(1+\epsilon)\tilde{u}(\xi,0)}{\epsilon+\nu\widetilde{F}(\xi,\epsilon^*)}, \tag{3.2}$$

where

$$\widetilde{F}(\xi,\epsilon^*)=\frac{\begin{array}{l}[\nu\ \mathrm{cosech}^2(\lambda\delta)(\nu-\lambda\coth\lambda)+\epsilon^*\{\lambda^2\coth^2\lambda\delta\\ +\quad(\nu^2-\lambda^2)(\coth\lambda\coth\lambda\delta)-\nu\lambda\coth\lambda\ \mathrm{cosech}^2(\lambda\delta)-\nu^2\}]\end{array}}{\begin{array}{l}\lambda[\nu\ \mathrm{cosech}^2(\lambda\delta)(\lambda-\nu\coth\lambda)+\epsilon^*\{\nu^2\coth(\lambda\delta)\times\\ (1-\coth\lambda\coth\lambda\delta)+\lambda^2(\coth\lambda-\coth\lambda\delta)\\ +\nu\lambda\mathrm{cosech}^2(\lambda\delta)\}]\end{array}}, \tag{3.3}$$

where we have defined $\lambda=\sqrt{k^2+\xi^2}$. Applying the Fourier inversion formula to the relation (3.2), we obtain

$$u_y(x,0)=\frac{\nu(1+\epsilon)}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{u}(\xi,0)e^{i\xi x}\ d\xi}{\epsilon+\nu\widetilde{F}(\xi,\epsilon^*)}\ \text{ on } y=0. \tag{3.4}$$

Then using the relations (2.25), (2.26), (2.27) and the condition (3.4), we find that the whole problem is reduced to that of determining the single unknown function $u(x,y)$, in the region $y<0$, satisfying the following equation and conditions:

$$\nabla^2 u=k^2u, \tag{3.5}$$

$$\frac{\partial u}{\partial n}=0,\ \text{ on } S, \tag{3.6}$$

and

$$u_y=\nu(1+\epsilon)Mu,\ \text{ on } y=0, \tag{3.7}$$

where

$$Mu=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{u}(\xi,0)e^{i\xi x}\ d\xi}{\epsilon+\nu\widetilde{F}(\xi,\epsilon^*)}, \tag{3.8}$$

and the function $\widetilde{F}(\xi,\epsilon^*)$ is as given by the relation (3.3). Additionally, we have the far field conditions (2.27) and (2.28).

4. Reduction to the spectral problem

Following Kuznetsov [4], we seek the function $u(x,y)$, satisfying the equation (3.5), of the following form

$$u(x,y)=(V\mu)(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\mu(\sigma)G(x,y;\sigma,0)\ d\sigma, \tag{4.1}$$

where $\mu(\sigma)$ is a square integrable function and the Green's function $G(x,y;\sigma,0)$ is given by the following relation (see Kuznetsov [4] and Ursell [2], for details):

$$\begin{aligned} G(x,y;\sigma,0) &= K_0(k[(x-\sigma)^2+y^2]^{\frac{1}{2}}) \\ &\quad + \int_S m(s,\sigma)\{K_0(k[(x-X(s))^2+(y-Y(s))^2]^{\frac{1}{2}}) \\ &\quad + K_0(k[(x-X(s))^2+(y+Y(s))^2]^{\frac{1}{2}})\}\,ds, \end{aligned} \tag{4.2}$$

where K_0 is the well known Macdonald function, s denotes the arc length along S, $X(s)$ and $Y(s)$ are the coordinates of a point on S, and the function $m(s,\sigma)$ is the unique solution of the following Fredholm integral equation of the second kind:

$$\begin{aligned} -\pi m(s,\sigma) + \int_S m(s',\sigma)\frac{\partial}{\partial n(s')}&\{K_0(k[(X(s')-X(s))^2+(Y(s')-Y(s))^2]^{\frac{1}{2}}) \\ &+ K_0(k[(X(s')-X(s))^2+(Y(s')+Y(s))^2]^{\frac{1}{2}})\}\,ds' \\ &= -\frac{\partial}{\partial n(s)}K_0(k[(X(s)-\sigma)^2+Y^2(s)]^{\frac{1}{2}}). \end{aligned} \tag{4.3}$$

It is easily verified that the function $u(x,y)$, given by the relation (4.1), satisfies the condition

$$u_y(x,0) = \frac{\partial}{\partial y}(V\mu) = \mu, \text{ when } y=0. \tag{4.4}$$

Substituting the relation (4.1) into the relation (3.7) and using Fourier transform, we obtain

$$\tilde{u}_y(\xi,0) = \tilde{\mu}(\xi) = \frac{\nu(1+\epsilon)\widetilde{V\mu}(\xi,0)}{\epsilon+\nu\widetilde{F}(\xi,\epsilon^*)}. \tag{4.5}$$

The above relation (4.5) can be cast into the following cubic spectral problem (after utilizing the Fourier inversion formula), involving the spectral parameter (or eigenvalue) ν and the eigenfunction μ, after setting $\epsilon^* = \alpha\epsilon$.

$$\begin{aligned} &\nu^3[A_0+\epsilon(A+\alpha A_1)+\alpha\epsilon^2 A_2]\mu - \nu^2\left[B+\epsilon(1+\alpha)B+\alpha\epsilon^2 B_1\right]\mu \\ &+\nu\left[\epsilon(C_0+\alpha C_1)+\alpha\epsilon^2(C_0+C_2)\right]\mu - \alpha\epsilon^2 D\mu = 0, \end{aligned} \tag{4.6}$$

where the operators $A_0, A_1, A_2, B, B_1, C_0, C_1, C_2$, and D are defined by the following relations:

$$\widetilde{A_0\mu} = \frac{1}{\lambda \sinh^2(\lambda\delta)} \left[\lambda^{-2}\tilde{\mu} + \lambda^{-1}\coth\lambda \widetilde{V\mu}\right], \tag{4.7}$$

$$\widetilde{A\mu} = \frac{\lambda^{-3}\coth\lambda}{\sinh^2(\lambda\delta)}\widetilde{V\mu}, \tag{4.8}$$

$$\widetilde{A_1\mu} = \lambda^{-3}(\coth(\lambda\delta)\coth\lambda - 1)\tilde{\mu} + \lambda^{-2}\coth(\lambda\delta)(\coth(\lambda\delta)\coth\lambda - 1)\widetilde{V\mu}, \tag{4.9}$$

$$\widetilde{A_2\mu} = \lambda^{-2}\coth(\lambda\delta)(\coth(\lambda\delta)\coth\lambda - 1)\widetilde{V\mu}, \tag{4.10}$$

$$\widetilde{B\mu} = \frac{1}{\lambda \sinh^2(\lambda\delta)}((\lambda^{-1}\coth\lambda)\tilde{\mu} + \widetilde{V\mu}), \tag{4.11}$$

$$\widetilde{B_1\mu} = \frac{1}{\lambda \sinh^2(\lambda\delta)}\widetilde{V\mu} + \lambda^{-2}\coth(\lambda\delta)(\coth\lambda\coth(\lambda\delta) - 1)\tilde{\mu}, \tag{4.12}$$

$$\widetilde{C_0\mu} = \frac{1}{\lambda \sinh^2(\lambda\delta)}\tilde{\mu}, \tag{4.13}$$

$$\widetilde{C_1\mu} = (\coth(\lambda\delta) - \coth\lambda)((\lambda^{-1}\coth(\lambda\delta))\tilde{\mu} + \widetilde{V\mu}), \tag{4.14}$$

$$\widetilde{C_2\mu} = (\coth(\lambda\delta) - \coth\lambda)\widetilde{V\mu}, \tag{4.15}$$

$$\widetilde{D\mu} = (\coth(\lambda\delta) - \coth\lambda)\tilde{\mu}. \tag{4.16}$$

In the next section, we analyze the spectral problem (4.6) by employing a perturbation approach which is similar to the one employed by Kuznetsov [4].

5. Perturbation method

By taking the inner-product of the equation (4.6), we obtain the cubic equation

$$\begin{aligned}\left(a_0 + \epsilon(a + \alpha a_1) + \alpha\epsilon^2 a_2\right)\nu^3 - \left(b + \epsilon(1+\alpha)b + \alpha\epsilon^2 b_1\right)\nu^2 \\ + \left(\epsilon(c_0 + \alpha c_1) + \alpha\epsilon^2(c_0 + c_2)\right)\nu - d\alpha\epsilon^2 = 0,\end{aligned} \tag{5.1}$$

for the determination of the eigenvalue ν where

$$\begin{gathered} a_0 =< A_0\mu, \mu >= \int_{-\infty}^{\infty} (A_0\mu)(x)\overline{\mu(x)}dx, \\ a =< A\mu, \mu >, \quad a_1 =< A_1\mu, \mu >, \quad a_2 =< A_2\mu, \mu >, \\ b =< B\mu, \mu >, \quad b_1 =< B_1\mu, \mu >, \\ c_0 =< C_0\mu, \mu >, \quad c_1 =< C_1\mu, \mu >, \quad c_2 =< C_2\mu, \mu >, \quad d =< D\mu, \mu > . \end{gathered}$$

The cubic relation (5.1), which holds good for small positive values of ϵ, implies just two possible forms of the perturbational expansion for the eigenvalue ν, along with the corresponding eigenfunctions μ, as given by

Form I:

$$\left.\begin{array}{l}\nu \equiv \nu^{+} = \nu_0 + \epsilon\nu_1 + \epsilon^2\nu_2 + \cdots \\ \mu \equiv \mu^{+} = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots\end{array}\right\} \tag{5.2}$$

and

Form II:

$$\left.\begin{array}{l}\nu \equiv \nu^{(0)} = \epsilon\nu_1^{(0)} + \epsilon^2\nu_2^{(0)} + \epsilon^3\nu_3^{(0)} + \cdots \\ \mu \equiv \mu^{(0)} = \mu_0^{(0)} + \epsilon\mu_1^{(0)} + \epsilon^2\mu_2^{(0)} + \cdots\end{array}\right\} \tag{5.3}$$

Now, on substituting the expansions (5.2) into the relation (4.6) and equating the various powers of ϵ on either side we arrive at the following set of relations:

$$\nu_0^3 A_0\mu_0 - \nu_0^2 B\mu_0 = 0, \tag{5.4}$$

$$\begin{aligned}\nu_0^3 A_0\mu_1 - \nu_0^2 B\mu_1 = {} & (1+\alpha)\nu_0^2 B\mu_0 + 2\nu_0\nu_1 B\mu_0 - \nu_0^3 A\mu_0 \\ & - \alpha\nu_0^3 A_1\mu_0 - 3\nu_0^2\nu_1 A_0\mu_0 - \nu_0(C_0+\alpha C_1)\mu_0,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\nu_0^3 A_0\mu_2 - \nu_0^2 B\mu_2 = {} & \alpha\nu_0^2 B_1\mu_0 + 2\nu_0\nu_1(1+\alpha)B\mu_0 + 2\nu_0\nu_1 B_1\mu_0 \\ & + \nu_1^2 B\mu_0 + (1+\alpha)\nu_0^2 B\mu_1 + 2\nu_0\nu_1 B\mu_1 - \alpha\nu_0^3 A_2\mu_0 \\ & - 3\alpha\nu_0^2\nu_1 A_1\mu_0 - 3\alpha\nu_0^2\nu_1 A\mu_0 - 3\nu_0\nu_1^2 A_0\mu_0 \\ & - 3\nu_0^2\nu_2 A_0\mu_0 - \nu_0^3 A\mu_1 - \alpha\nu_0^3 A_1\mu_1 - 3\nu_0^2\nu_1 A_0\mu_1 + \\ & + \alpha D\mu_0 - \nu_1(C_0+\alpha C_1)\mu_0 - \alpha\nu_0(C_0+C_2)\mu_0 \\ & - \nu_0(C_0+\alpha C_1)\mu_1,\end{aligned} \tag{5.6}$$

$$\begin{aligned}\nu_0^3 A_0\mu_3 - \nu_0^2 B\mu_3 = {} & 2\alpha\nu_0\nu_1 B_1\mu_0 + (1+\alpha)\nu_1^2 B\mu_0 \\ & + 2\nu_0\nu_2(1+\alpha)B\mu_0 + 2\nu_1\nu_2 B\mu_0 + 2\nu_0\nu_3 B\mu_0 \\ & + \alpha\nu_0^2 B\mu_1 + 2\nu_0\nu_1(1+\alpha)B\mu_1 + 2\nu_1\nu_0 B_1\mu_1 \\ & + \nu_1^2 B\mu_1 + (1+\alpha)\nu_0^2 B\mu_2 + 2\nu_0\nu_1 B\mu_2 \\ & - 3\alpha\nu_0^2\nu_1 A_2\mu_0 - 3\alpha\nu_0\nu_1^2 A_1\mu_0 - 3\alpha\nu_0^2\nu_2 A_1\mu_0 \\ & - 3\nu_0\nu_1^2 A\mu_0 - 3\nu_0^2\nu_2 A\mu_0 - \nu_1^3 A_0\mu_0 \\ & - 3\nu_0^2\nu_3 A_0\mu_0 - \alpha\nu_0^3 A_2\mu_1 - 3\alpha\nu_0^2\nu_1 A_1\mu_1 \\ & - 3\alpha\nu_0^2\nu_1 A\mu_1 - 3\nu_0\nu_1^2 A_0\mu_1 - 3\nu_0^2\nu_2 A_0\mu_1 \\ & - 3\nu_0\nu_1^2 A_0\mu_1 - 3\nu_0^2\nu_2 A_0\mu_1 + \nu_0^3 A\mu_2 \\ & - \alpha\nu_0^3 A_1\mu_2 - 3\nu_0^2\nu_1 A_0\mu_2 \\ & + \alpha D\mu_1 - \nu_2(C_0+\alpha C_1)\mu_0 - \alpha\nu_1(C_0+C_2)\mu_0 \\ & - \nu_1(C_0+\alpha C_1)\mu_1 - \alpha\nu_0(C_0+C_2)\mu_1,\end{aligned} \tag{5.7}$$

and so on.

Similarly, substituting the expansions (5.3) into the relation (4.6) and equating the various powers of ϵ on either side leads to the following set of relations:

$$\nu_1^{(0)^2} B\mu_0^{(0)} - \nu_1^{(0)}(C_0 + \alpha C_1)\mu_0^{(0)} + \alpha D\mu_0^{(0)} = 0, \tag{5.8}$$

$$\begin{aligned} &\nu_1^{(0)^2} B\mu_1^{(0)} - \nu_1^{(0)}(C_0 + \alpha C_1)\mu_1^{(0)} + \alpha D\mu_1^{(0)} = -\nu_1^{(0)^2}(1+\alpha)B\mu_0^{(0)} \\ &-2\nu_1^{(0)}\nu_2^{(0)} B\mu_0^{(0)} + \nu_1^{(0)^3} A_0\mu_0^{(0)} + \nu_2^{(0)}(C_0 + \alpha C_1)\mu_0^{(0)} + \alpha\nu_1^{(0)}(C_0 + C_2)\mu_0^{(0)} \end{aligned} \tag{5.9}$$

and so on.

The first of the set of relations (5.4)-(5.7) gives rise to the eigenvalue problem

$$\nu_0 A_0\mu_0 - B\mu_0 = 0, \tag{5.10}$$

for the determination of the nonzero eigenvalue ν_0 and the eigenfunction μ_0, where the operators A_0 and B are as defined by the relations (4.7) and (4.11). The eigenvalue problem (5.10) is the same as the one encountered by Kuznetsov [4] in which these operators are defined as:

$$A_0 = LV + C, \quad \text{and } B = L + V, \tag{5.11}$$

where

$$C\mu(x) = (2k)^{-1}\int_{-\infty}^{\infty} e^{-k|x-\sigma|}\mu(\sigma)\, d\sigma, \tag{5.12}$$

$$L\mu(x) = \int_{-\infty}^{\infty} \mu(\sigma)\left(\frac{1}{\pi}\int_0^{\infty} \frac{\coth\lambda}{\lambda}\cos(\xi|x-\sigma|)\, d\xi\right) d\sigma, \tag{5.13}$$

and

$$V\mu(x) = V\mu(x,0),$$

with the operator V defined by the relation (4.1).

It has been shown by Kuznetsov [4] that the eigenvalue ν_0, associated with (5.10), is positive which is, therefore, also the case in the present problem for all values of the parameter δ involving the thickness of the top layers of fluids under our consideration. Then having determined the value of ν_0 and the corresponding eigenfunction μ_0 using arguments similar to one in [4], we can determine the values of $\nu_1, \mu_1, \nu_2, \mu_2$, etc. by applying the solvability criteria to the inhomogeneous equations (5.4), (5.5), (5.6), (5.7), etc. It can be shown ultimately that the eigenvalue ν in the expansion (5.2) is a positive number for a sufficiently small value of the parameter ϵ.

We have thus shown that there exists a set of positive numbers ν for sufficiently small values of the parameter $\epsilon = \frac{\rho}{\rho^*} - 1$ (see the relations (2.14)), assuring the existence of trapped modes of waves on the free surface for all positive values of the parameters δ and α.

In contrast with the set of relations (5.4)-(5.7) which correspond to the Form-I of the unknowns ν and μ, we find that the set of relations (5.8)-(5.9), corresponding to the Form-II of these unknowns, produce two distinct solutions μ given by the expansion (5.3) for any arbitrary positive value of $\delta < 1$. This fact is obvious from relations (4.7)-(4.16) and the positivity of all the operators B, C_0, C_1 and D appearing in these relations. In support of this statement, we note the following.

Using inner product along with the relation

$$< P\mu(x), \mu(x) >= \frac{1}{2\pi} < \widetilde{P\mu}(\xi), \tilde{\mu}(\xi) >\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{p}(\xi)|\tilde{\mu}(\xi)|^2 d\xi,$$

for any operator P of the form

$$P\mu(x) = \int_{-\infty}^{\infty} p(x-\sigma)\mu(\sigma)\, d\sigma,$$

where $\tilde{\mu}(\xi)$ represents the Fourier transform of $\mu(x)$, we find that relation (5.9) can be cast into the form

$$p_0(q_0 + \tilde{v})\nu_1^{(0)^2} - [p_0 + c_0(r_0 + \tilde{v})]\, \nu_1^{(0)} + c_0 = 0, \tag{5.14}$$

where

$$\begin{aligned} p_0 &= \frac{1}{\lambda \sinh^2(\lambda\delta)}, \quad q_0 = \lambda^{-1}\coth\lambda, \\ r_0 &= \lambda^{-1}\coth(\lambda\delta), \quad c_0 = \alpha(\coth(\lambda\delta) - \cot(\lambda)), \end{aligned} \tag{5.15}$$

and

$$\tilde{v}(\xi) = \int_{-\infty}^{\infty} v(\sigma)e^{i\xi\sigma}\, d\sigma. \tag{5.16}$$

It is worth emphasizing that it can be shown that $V\mu(x)$ defined before is related to $v(\sigma)$ through the following convolution.

$$V\mu(x) = \int_{-\infty}^{\infty} v(x-\sigma)\mu(\sigma)\, d\sigma. \tag{5.17}$$

Then, using the fact that operator V is positive and $r_0 > q_0$ for $0 < \delta < 1$, we find that both the roots of the quadratic equation (5.14) are positive.

We thus conclude that three sets of frequencies of trapped modes exist in the case of the hydrodynamic set-up, comprising of three layers of fluids, as considered in the present work, which is a more general result as compared to the set-up of Kuznetsov [4], involving just two layers of fluids.

5.1. Case studies for extreme and limiting values of the parameters

In this section we discuss within the framework of the analysis given in the previous section the zero-parameter solution (e.g., $\epsilon = 0$) and the zero-parameter-limit (e.g., $\epsilon \to 0$) solution for each of the two parameters, ϵ and α separately.

Two-layer fluid limit: If $\alpha = 0$ with $\epsilon > 0$, then $\epsilon^* = \alpha\epsilon = 0$ meaning $\rho^* = \rho^{**}$ (see (2.14)). Therefore, in this case, the upper internal interface disappears reducing the problem to the two-layer case which has been studied by Kuznetsov [4] using perturbation approach and also very recently by Linton and Cadby [5] (refereed as 'LC' henceforth) using completely numerical approach (we comment more on LC later). These studies have shown that the solution of this problem supports trapped modes on both the free surface and the interface for finite density-difference between the two layers.

Now we look at the zero-α-limit (i.e. $\alpha \to 0$) solution of our three-layer fluid problem. In the limit $\alpha \to 0\,(\epsilon > 0)$, equation (4.6) reduces to

$$\nu^2(A_0 + \epsilon A)\mu - \nu(1+\epsilon)B\mu + \epsilon\, C_0\mu = 0.$$

Then taking its Fourier transform and using (4.7), (4.8), (4.11), and (4.13), it is easy to compare the resulting relation with the Fourier transformed version of the relation (22) of Kuznetsov [4] and then conclude that our results in the limiting case $\alpha \to 0$ reduce to those of Kuznetsov [4]. Thus, two-layer-limit solution of the three-layer problem considered here is same as the two-layer solution of Kuznetsov [4]. In this case, it is interesting and important to observe from equation (5.14) that there exists only one positive value of $\nu_1^{(0)}$ corresponding to the trapped mode on the interface of two fluid layers. Therefore, the trapped modes $\nu_1^{(0)}$ that exist in this limit must correspond to the trapped modes on the lower interface of the three-layer fluid set-up. Similarly, the trapped modes $\nu_1^{(0)}$ that disappears in this limit must correspond to the trapped modes on the upper interface of the three-layer fluid set-up.

Single-layer fluid limit: We consider here single-layer-limit solution of our three-layer problem. In our three-layer-fluid problem if $\epsilon = 0$, then $\epsilon^* = \alpha\epsilon = 0$ and (2.14) implies that there is only a single layer of fluid of infinite depth which is the problem considered in Ursell [2]. There it has been shown that the solution of this problem supports trapped modes on the free surface. This has been mentioned before in the section on Introduction.

Now we look at the zero-ϵ-limit (i.e. $\epsilon \to 0$) solution of our three-layer fluid problem. In the limit $\epsilon \to 0$ $(\alpha > 0)$, equation (4.6), after setting $\epsilon = 0$ and then taking its Fourier transform, reduces to

$$\nu A_0\tilde{\mu} - B\tilde{\mu} = 0.$$

Using the definitions (4.7) and (4.11), this equation can be cast into

$$\nu(\lambda^{-2}\tilde{\mu} + q_0\widetilde{V\mu}) - (q_0\tilde{\mu} + \widetilde{V\mu}) = 0,$$

where q_0 is given by (5.15). Using inner product and the convolution theorem of Fourier transform along with the fact that V is a positive operator, we can show, after using the fact that $\tilde{v} > 0$ which follows from (5.16), that

$$\nu = \frac{q_0 + \tilde{v}}{\lambda^{-2} + q_0\tilde{v}} > 0,$$

and this establishes the Ursell's result [2] that positive values of ν exist in the single fluid case of our hydrodynamic set-up. Also note that the eigenvalue ν^+

defined in (5.2) reduces to ν_0 and the eigenvalues $\nu^{(0)}$ defined through (5.3) vanish in this limit, Therefore, this eigenvalue ν^+ corresponds to trapped modes on the free surface of the three-layer fluid set-up.

Previously, Kuznetsov (see [4]) has shown using the same perturbation approach on two-layer fluid problem (as opposed to three-layer case considered here) that his two-layer solution also recovers Ursell's result [2] in the single-layer limit.

Remarks: Some remarks are necessary here in light of very recent numerical work of Linton and Cadby [5] (refereed as 'LC' henceforth) on trapped modes in two-layer fluid. This recent work of LC was brought to our attention after completion of our work presented above. This two-layer problem which was originally solved by Kuznetsov a decade ago using perturbation approach as mentioned in this paper, has been recently solved by LC using numerical approach, but for arbitrary density ratio between two layers. The single-layer-limit solution of LC, obtained numerically by extrapolating numerical data for finite-density-difference to zero-density-difference between layers, supports trapped modes only on the interface (of the limit problem) and none on the free surface. Therefore, the single-layer-limit solution of the two-layer **numerical** solution of LC is not same as that of the two-layer **perturbation** solution of Kuznetsov. In LC, no mathematically sound resolution of this paradox (i.e. the above mentioned contradictory results between perturbation approach of Kuznetsov and numerical approach of LC) has been provided. LC provides a physical interpretation of their limit solution which is that that in the presence of a second layer, however small its width may be, the trapped modes on the free-surface starts leaking energy along the interface as suggested by LC (see also LC [5]).

What all this means for our three-layer problem can be a speculation at best which is this. If one were to solve our three-layer-problem using LC's numerical approach (assuming that his numerical results are not flawed), then the single-layer-limit solution of this three-layer numerical solution should perhaps support only the modes associated with the lowest interface which is not what we get using perturbation approach as discussed before. If all this were true, then it would mean that the single-layer-limit solution of the three-layer **numerical** solution is not same as that of the three-layer **perturbation** solution presented here. At this point, we do not have any mathematically rigorous resolution of the difficulties mentioned here since it first requires solving these problems numerically to test the validity of existing numerical results for the two-layer case and to obtain numerical results for the three-layer case which have not been worked out yet by anyone. All this falls outside the scope of this paper and perhaps will be addressed in the future.

6. Discussions and conclusions

The mathematical analysis involving a perturbation technique applied to a polynomial-cum-operator eigenvalue problem shows here the existence of trapped modes of waves in three layers of fluids in a horizontal channel containing a fully immersed cylinder in the bottom layer. This generalizes the results of Kuznetsov [4] from two layer case to three layer case. It has been shown that whilst two definite sets of frequencies of trapped modes exist in the case of two layers, there exist three sets of such trapped modes in the case of three fluid layers. The one set corresponds to the trapped modes of waves on the free surface and the other two sets of frequencies correspond to the two internal interfacial modes of trapped waves, one for each of the internal interfaces. An analysis of the correlation between our results for the three-layer case and very recent numerical results of LC in the two-layer case is given and some inconsistencies in the limiting cases between the perturbation and numerical solutions are pointed out.

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A. Chakrabarti
Department of Mathematics
Indian Institute of Science
Bangalore 560012
India

P. Daripa
Department of Mathematics
Texas A&M University
College Station
TX-77843
USA
e-mail: prabir.daripa@math.tamu.edu

Hamsapriye
University of Cambridge
Department of Haematology
Cambridge Institute of Medical Research
Wellcome Trust / MRCBuilding
Hills Road
Cambridge CB2 2XY
U.K.

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