Chapter 4

Local and pointwise error estimates

In this chapter we make a brief excursion into the area of local and pointwise (maximum-norm) error estimates for the finite element method. We shall emphasize a priori error estimates. In keeping with the spirit of the previous chapters, we will pay careful attention to domain regularity in our estimates. Another issue which arises when proving error estimates in norms other than the global energy norm is mesh regularity. It is straightforward to prove optimal a priori error estimates in the energy norm on shape-regular grids, and in particular on adaptively-generated grids. The situation for other norms has proved to be much more difficult, and we will highlight this issue in our discussion as well.

4.1 Local energy estimates

Local energy estimates are both interesting in their own right and useful as a tool for proving maximum-norm error estimates. We begin with motivation from analysis, then establish an important technical tool (superapproximation), and finally proceed to prove local error estimates.

4.1.1 Caccioppoli inequalities

It is standard intuition that one cannot bound a stronger norm of a function by a weaker norm, but this intuition turns out to be wrong (or at least incomplete) for harmonic functions.

**Proposition 4.1.1** Let \(-\Delta u = 0\) on \(\Omega\), with \(u \neq 0\) on \(\partial \Omega\). Let also \(D \subset \Omega\) and \(d > 0\) be given such that \(D_d := \{x \in \Omega : \text{dist}(x, D) \leq d\} \subset \subset \Omega\). Then

\[
\|\nabla u\|_{L^2(D)} \lesssim d^{-1} \|u\|_{L^2(D_d)}. \tag{4.1}
\]

**Proof.** Let \(\omega \in C_0^\infty(D_d)\) be a cutoff function satisfying \(\omega \equiv 1\) on \(D\) and \(\|D^m \omega\|_{L^\infty(D_d)} \lesssim d^{-m}, m \geq 0\). Then

\[
\|\nabla u\|_{L^2(D)}^2 \leq \|\omega \nabla u\|_{L^2(\Omega)}^2 = \int_\Omega \omega^2 \nabla u \cdot \nabla u
\]

\[
= \int_\Omega \nabla u \cdot \nabla (\omega^2 u) - 2 \int_\Omega (\omega \nabla u) \cdot (u \nabla \omega) \tag{4.2}
\]
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But \( \int_T \nabla u \cdot \nabla (\omega^2 u) = 0 \) because \( u \) is harmonic. Applying Hölder’s inequality, we thus obtain

\[
\|\nabla \omega u\|_{L^2(\Omega)}^2 \leq 2\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(D_\delta)}\|\nabla \omega\|_{L^2(\Omega)} \lesssim d^{-1}\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(D_\delta)}.
\]

Dividing through by \( \|\nabla \omega u\|_{L^2(\Omega)} \) completes the proof. \( \square \)

4.1.2 Superapproximation

Superapproximation is an essential tool in local and pointwise error analysis for FEM. It was introduced in the classical paper [41] in which local error estimates were first proved. We give a slightly sharper version from [22] which will allow us to prove local error estimates on general shape regular grids.

**Lemma 4.1.2** Assume that \( T \) is a shape-regular simplex in \( \mathbb{R}^d \), and that \( \omega \in C^\infty(T) \) satisfies

\[
\|D^m \omega\|_{L^\infty(T)} \lesssim d^{-m}, \quad m = 0, \ldots, r + 1 \text{ with } d \geq h_T := \text{diam}(T).
\]

Let also \( I_h \) be the Lagrange interpolant on \( T \) into \( \mathbb{P}^r \). Then for \( \chi \in \mathbb{P}^r \), there holds

\[
\|\omega^2 \chi - I_h(\omega^2 \chi)\|_{H^1(T)} \lesssim \frac{h_T}{d} \|\nabla \omega\|_{L^2(\Omega)} \|
abla \chi\|_{L^2(\Omega)} + \frac{h_T^2}{d^2} \|\chi\|_{L^2(\Omega)}. \tag{4.3}
\]

**Proof.** We use Hölder’s inequality and standard approximation theory to find

\[
\|\omega^2 \chi - I_h(\omega^2 \chi)\|_{H^1(T)} \leq C h_T^{r/2} \|\omega^2 \chi - I_h(\omega^2 \chi)\|_{W^1_{\infty}(T)} \leq C h_T^{r/2 + 1} |\omega^2 \chi|_{W^r_{\infty}(T)}. \tag{4.4}
\]

Noting that \( D^\alpha \chi = 0 \) for all multiindices \( \alpha \) with \( |\alpha| = r + 1 \), recalling that \( \frac{h_T^2}{d^2} \leq 1 \), and employing inverse estimates, we compute

\[
h_T^{n/2 + r} |\omega^2 \chi|_{W^r_{\infty}(T)} \lesssim h_T^{n/2 + r} \sum_{i=1}^{r+1} \sum_{|\alpha|=i, |\beta|=r+1-i} \|D^\alpha \omega^2 D^\beta \chi\|_{L^\infty(T)}
\]

\[
\lesssim (\sum_{i=2}^{r+1} h_T^{i-1} |\omega^2|_{W^r_{\infty}(T)}) \|
abla \chi\|_{L^2(\Omega)} + h_T^{n/2 + r} \sum_{|\alpha|=1, |\beta|=r} \|D^\alpha \omega^2 D^\beta \chi\|_{L^\infty(T)}
\]

\[
\lesssim \frac{h_T}{d^2} \|\nabla \chi\|_{L^2(\Omega)} + \frac{h_T^{n/2 + r}}{d^2} \sum_{|\alpha|=1, |\beta|=r} \|D^\alpha \omega^2 D^\beta \chi\|_{L^\infty(T)}. \tag{4.5}
\]

We next consider the terms \( \|D^\alpha \omega^2 D^\beta \chi\|_{L^\infty(T)} \). Since \( |\alpha| = 1 \), we have \( D^\alpha \omega^2 = 2\omega D^\alpha \omega \). Let \( \tilde{\omega} = \frac{1}{|T|} \int_T \omega \, dx \) so that \( \|\omega - \tilde{\omega}\|_{L^\infty(T)} \lesssim h_T \|\omega\|_{W^1_{\infty}(T)} \lesssim \frac{h_T}{d^2} \). Employing inverse estimates, we thus
have
\[
\begin{align*}
&h_T^{-n/2} \sum_{|\alpha|=1,|\beta|=r} \|D^\alpha \omega^2 D^\beta \chi\|_{L^\infty(T)} \\
&\leq d^{-1} h_T^{-n/2} \sum_{|\beta|=r} \|\omega D^\beta \chi\|_{L^\infty(T)} \\
&\leq d^{-1} h_T^{-n/2} \sum_{|\beta|=r} (\|\omega - \hat{\omega}\|_{L^\infty(T)} + \|\hat{\omega} D^\beta \chi\|_{L^\infty(T)} ) \\
&\leq (\frac{h_T}{d})^2 \|\chi\|_{L^2(T)} + \frac{h_T}{d} \|\hat{\omega} \chi\|_{H^1(T)} \\
&\leq (\frac{h_T}{d})^2 \|\chi\|_{L^2(T)} + \frac{h_T}{d} \|\hat{\omega} \chi\|_{H^1(T)} + \frac{h_T}{d} \|\omega \chi\|_{H^1(T)}. 
\end{align*}
\]  

Using an inverse inequality, we find that
\[
\frac{h_T}{d} (\hat{\omega} - \omega) \chi |_{H^1(T)} \leq \frac{h_T}{d} (\|\omega |_{W^1_2(T)}\|_{L^2(T)} + \|\hat{\omega} - \omega\|_{L^\infty(T)} \|\chi\|_{H^1(T)}) \\
\leq C \frac{h_T}{d} (\frac{1}{d} \|\chi\|_{L^2(T)} + \frac{h_T}{d} \|\chi\|_{H^1(T)}) \\
\leq C \frac{h_T}{d} \|\chi\|_{L^2(T)}. 
\]

Inserting (4.7) into (4.6) and the result into (4.5) and (4.4) completes the proof. \(\Box\)

### 4.1.3 Local energy estimates

We now prove a local error estimate. Such estimates were first proved in [41] and improved to allow general shape-regular (as opposed to quasi-uniform) meshes in [22]. As an aside, the local energy estimate in [22] is the only error estimate we are aware of for finite element methods in a norm other than a global energy norm that is valid on arbitrary shape-regular grids.

**Theorem 4.1.3** Assume that \(T_h\) is a shape-regular simplicial decomposition of a domain \(\Omega \subset \mathbb{R}^d\), and that \(S_h\) is a Lagrange finite element space of degree \(r\) on \(T_h\). Let also \(D \subset \Omega\) be given, and given \(d > 0\) let \(D_d := \{x \in \Omega : \text{dist}(x, D) < d\}\). Assume that \(Kh_T \leq d\) for each \(T \in T_h\) with \(T \cap D_d \neq \emptyset\) and \(K\) sufficiently large. Finally, let \(u \in H^1(\Omega)\), and assume that \(u_h \in S_h\) satisfies
\[
\int_{D_d} \nabla u \nabla v_h = \int_{\Omega} \nabla u_h \nabla v_h, \quad v_h \in S_h, v_h = 0 \text{ on } \Omega \setminus D_d. 
\]

Then
\[
\|\nabla(u - u_h)\|_{L^2(D)} \leq \inf_{\chi \in S_h} \left( \|\nabla (u - \chi)\|_{L^2(D_d)} + \frac{1}{d} \|u - \chi\|_{L^2(D_d)} \right) + \frac{1}{d} \|u - u_h\|_{L^2(D_d)}. \tag{4.8}
\]

We discuss the bound (4.8). The first two terms \(\|\nabla (u - \chi)\|_{L^2(D_d)} + \frac{1}{d} \|u - \chi\|_{L^2(D_d)}\) are local almost-best-approximation terms; we could not expect much better of the local error behavior. The last term \(\frac{1}{d} \|u - u_h\|_{L^2(D_d)}\) is a “pollution” term which expresses influence of the global solution...
Similarly employing Young’s inequality on the terms Young’s inequality to find for

\[ \| \nabla (u_h - \chi) \|^2_{L^2(D)} \leq \| \omega \nabla (u_h - \chi) \|^2_{L^2(D)} \]

\[ = \int_\Omega \nabla (u_h - \chi) \nabla [\omega^2 (u_h - \chi)] - 2 \int_\Omega \omega \nabla (u_h - \chi) [(u_h - \chi) \nabla \omega] \]

\[ \leq \int_\Omega \nabla (u_h - \chi) \nabla [\omega^2 (u_h - \chi)] \]

\[ + \frac{1}{d} \| \omega \nabla (u_h - \chi) \|_{L^2(D_d)} \| u_h - \chi \|_{L^2(D_d)} \]

\[ := I + II \]

The assumption that \( \omega \) is 0 outside of \( D_{d/2} \) and that \( d \geq Kh \) with \( K \) sufficiently large ensures that \( I_h (\omega^2 (u_h - \chi)) \) is 0 outside of \( D_d \). We may thus apply Galerkin orthogonality to obtain

\[ I = \int_\Omega \nabla (u_h - u) \nabla [\omega^2 (u_h - \chi)] + \int_\Omega \nabla (u - \chi) \nabla [\omega^2 (u_h - \chi)] \]

\[ \leq \int_\Omega \nabla (u_h - u) \nabla [\omega^2 (u_h - \chi) - I_h (\omega^2 (u_h - \chi))] \]

\[ + \| \nabla (u - \chi) \|_{L^2(D_d)} \frac{1}{d} \| u_h - \chi \|_{L^2(D_d)} + \| \omega \nabla (u_h - \chi) \|_{L^2(D_d)} \]

\[ := I_a + I_b. \]

We finally employ the superapproximation estimate (4.3), an inverse estimate, \( h_T/d \lesssim 1 \), and Young’s inequality to find for \( \epsilon > 0 \)

\[ I_a \leq \sum_{T \cap D_d \neq \emptyset} \| \nabla (u - u_h) \|_{L^2(T)} h_T \| (d^{-1} \| \omega \nabla (u_h - \chi) \|_{L^2(T)} + d^{-2} \| u_h - \chi \|_{L^2(T)}) \]

\[ \lesssim \sum_{T \cap D_d \neq \emptyset} \| \nabla (u - u_h) \|_{L^2(T)} \| \nabla (u - u_h) \|_{L^2(T)} h_T \| (d^{-1} \| \omega \nabla (u_h - \chi) \|_{L^2(T)} + d^{-2} \| u_h - \chi \|_{L^2(T)}) \]

\[ \lesssim \sum_{T \cap D_d \neq \emptyset} \| \nabla (u - u_h) \|_{L^2(T)} \| \nabla (u - u_h) \|_{L^2(T)} + \| \nabla (u - u_h) \|_{L^2(T)} + d^{-1} \| u_h - \chi \|_{L^2(T)} \]

\[ \lesssim \epsilon^{-1} \| \nabla (u - \chi) \|^2_{L^2(D_d)} + d^{-2} \| u - u_h \|^2_{L^2(D_d)} + \| u - \chi \|^2_{L^2(D_d)} + \| \omega \nabla (u_h - \chi) \|^2_{L^2(D_d)} \]
4.2. \(L_\infty\) error estimates on smooth domains

We next prove sharp pointwise error estimates on smooth domains, following the work of Schatz [44]. From these we can easily recover standard maximum-norm error estimates.

4.2.1 Statement of results

We assume that \(\Omega \subset \mathbb{R}^d\) is given with \(\partial \Omega\) smooth, and that \(u \in H^1(\Omega)\) satisfies

\[
L(u, v) := \int_{\Omega} A \nabla u \nabla v + b \cdot \nabla u v + c u v = \int_{\Omega} f v =: (f, v), \quad \forall v \in H^1(\Omega)
\]

for a coercive and continuous bilinear form \(L\) with smooth coefficients \(A, b, c\). Here natural boundary conditions \(A \nabla u \cdot \vec{n} = 0\) are enforced implicitly.

Our FEM is defined as follows. We assume that \(S_h\) is a quasi-uniform mesh (NOT shape-regular; we have significantly restricted our choice of meshes!) of mesh width \(h\) and corresponding Lagrange finite element space \(S_h\) of degree \(r\). The finite element solution is the unique \(u_h \in S_h\) satisfying \(L(u_h, v_h) = (f, v_h)\) for all \(v_h \in S_h\). Because \(\partial \Omega\) is smooth, this means that some “triangles” have a curved face, so it is not obvious that standard approximation properties, etc., automatically hold.

We will however assume that all needed properties (approximation estimates, inverse estimates, superapproximation) are valid even on boundary elements. We fix a point \(x_0 \in \Omega\) at which the error is to be measured and finally define a weight

\[
\sigma_{x_0,h}(x) = \frac{h}{|x-x_0|+h}.
\]

Theorem 4.2.1 Let the assumptions on the solution \(u\), finite element approximation \(u_h\), and finite element space \(S_h\) given above hold, and let \(0 \leq s \leq r - 1\). Then for \(x_0 \in \Omega\)

\[
|(u - u_h)(x_0)| \lesssim h(\ln \frac{1}{h})^{\delta_{r-s-1}} \inf_{\chi \in S_h} \|\sigma_{x_0,h}(u - \chi)\|_{L_{\infty}(\Omega)} + \|\sigma_{x_0,h} \nabla (u - \chi)\|_{L_{\infty}(\Omega)} \tag{4.14}
\]

Thus

\[
\|u - u_h\|_{L_{\infty}(\Omega)} \lesssim h(\ln \frac{1}{h})^{\delta_{r-s}} \inf_{\chi \in S_h} \|u - \chi\|_{W_{s}^r(\Omega)} \lesssim h^r (\ln \frac{1}{h})^{\delta_{r-s}} |u|_{W_{s}^{r+1}(\Omega)}. \tag{4.15}
\]

The estimate (4.15) is contained in a number of previous papers going back to the 1970’s under various assumptions on the regularity of \(\partial \Omega\). (4.14) is a “finer” estimate in that it measures the error at a point instead of in the maximum norm and expresses a more local dependence of the error at \(x_0\) on the properties of \(u\). More precisely, when \(r > 1\) (piecewise quadratic and higher-degree elements), we may take \(s > 0\) in (4.14). In this case the approximation error at points \(x\) with \(|x - x_0| \approx 1\) is multiplied by \(\sigma_{x_0,h}(x_0) \approx h^s\), and we have \(|\sigma_{x_0,h}(x) \nabla (u - \chi)(x)| \leq h^{r+s}\). On the other hand, at \(x_0\) we have \(\sigma_{x_0,h}(x_0) = 1\) and thus here \(\sigma_{x_0,h}(x_0) \nabla (u - \chi)(x_0) \approx h^r\) in general. That is, the weight \(\sigma\) de-emphasizes the influence of the approximability of \(u\) at points removed from the target point \(x_0\).
4.2.2 Proof

The proof of (4.14) is rather involved; for the sake of time we shall omit some details.

Step 1: Error representation by a regularized Green’s function. Given \( x_0 \in \Omega \), there is a smooth regularized \( \delta \)-function \( \delta_{x_0}^{\delta} \) supported in the element \( T \in T_h \) containing \( x_0 \) such that

\[
(v_h, \delta_{x_0}^{\delta}) = v_h(x_0), \quad v_h \in S_h,
\]

and

\[
\|\delta_{x_0}^{\delta}\|_{W^{p,m}(\Omega)} \lesssim h^{-d(1-1/p)-m}.
\]

Thus for \( \chi \in S_h \),

\[
|(u - u_h)(x_0)| \leq |(u - \chi)(x_0)| + |(\chi - u_h, \delta_{x_0}^{\delta})| \\
\leq |(u - \chi)(x_0)| + |(\chi - u, \delta_{x_0}^{\delta})| + |(u - u_h, \delta_{x_0}^{\delta})| \\
\lesssim \|u - \chi\|_{L_\infty(T_{x_0})} + |(u - u_h, \delta_{x_0}^{\delta})|.
\]

The first term above is bounded by \( C\|\sigma_{x_0,h}(u - \chi)\|_{L_\infty(\Omega)} \), so we are left to bound the second.

Step 2: Dual representation by a regularized Green’s function. Let \( g \in H^1(\Omega) \) satisfy \( L(u, g) = (v, \delta_{x_0}^{\delta}) \) for all \( v \in H^1(\Omega) \). Let also \( g_h \in S_h \) satisfy \( L(v_h, g_h) = L(v, g) \), \( v_h \in S_h \). Using Galerkin orthogonality yields

\[
(u - u_h, \delta_{x_0}^{\delta}) = L(u - u_h, g) = L(u - \chi, g - g_h) \\
\leq (\|\sigma_{x_0,h}^{\delta}(u - \chi)\|_{L_\infty(\Omega)} + \|\nabla(u - \chi)\|_{L_\infty(\Omega)}(4.19) \\
\times (\|\sigma_{x_0,h}^{\delta}(g - g_h)\|_{L_1(\Omega)} + \|\nabla(g - g_h)\|_{L_1(\Omega)})
\]

We thus must show that

\[
\|\sigma_{x_0,h}^{\delta}(g - g_h)\|_{L_1(\Omega)} + \|\nabla(g - g_h)\|_{L_1(\Omega)} \lesssim h^{\delta,\gamma-1}.
\]

Step 3: Dyadic decomposition and local energy bounds. We first decompose \( \Omega \) into dyadic annuli about \( x_0 \). Let \( M \) be a positive constant which we shall eventually take to be sufficiently large, and let \( d_j := 2^j Mh, j = 0, 1, ..., J \) with \( J \) the smallest integer so that \( d_J > \text{diam}(\Omega) \). Let

\[
\Omega_0 := B_{Mh}(x_0), \quad \Omega_j := \{x \in \Omega : d_{j-1} \leq |x - x_0| < d_j\}, \quad j \geq 1.
\]

Then \( \Omega = \bigcup_{j=0}^J \Omega_j \). Note also that \( \sigma_{x_0,h}(x) \sim h^{-j}d_j \) for \( x \in \Omega_j, j \geq 1 \). We also define

\[
\Omega_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}, \quad \Omega_j' = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}, \quad \text{etc.}
\]

Using a local estimate such as (4.8), we then compute for any \( \psi \in S_h \)

\[
\|\sigma_{x_0,h}^{\delta}(g - g_h)\|_{L_1(\Omega)} + \|\nabla(g - g_h)\|_{L_1(\Omega)} \lesssim h^{\delta,\gamma-1}
\]

\[
\leq \sum_{j=0}^J (h/d_j)^s d_j^{d_j/2} \|g - g_h\|_{H^s(\Omega_j)}
\]

\[
\lesssim \sum_{j=0}^J (h/d_j)^s d_j^{d_j/2} (\|\nabla(g - \psi)\|_{L_2(\Omega_j)} + d_j^{-1}(\|g - \psi\|_{L_2(\Omega_j)} + d_j^{-1}\|g - g_h\|_{L_2(\Omega_j)}).
\]

...
We next use the Green’s function (not the regularized variety), which satisfies \( L(G^{x_0}, v) = v(x_0) = (G^{x_0}, L v) \) for any \( x_0 \in \Omega \), where \( L(w, v) = (w, L v), w \in H^1(\Omega) \). It is known that if all problem data are smooth,

\[
D_{x_0}^\alpha D_x^\beta G^{x_0}(x) \lesssim |x - x_0|^{2-d-|\alpha|-|\beta|}, \quad |\alpha| + |\beta| \geq 1.
\]

Thus for \( y \in \Omega_j \) (\( j \geq 2 \)) and \( |\alpha| = r + 1 \),

\[
D^\alpha g(y) = D_y^\alpha (G^y, \delta^{x_0}) = (D_y^\alpha G^y, \delta^{x_0})
\]

\[
\lesssim \|D_y G^y\|_{L^\infty(T_{x_0})} \|\delta^{x_0}\|_{L^1(T_{x_0})} \lesssim d_j^{2-d-(r+1)}.
\]

Applying approximation properties and recalling that \( h/d_j \leq 1 \), we thus have for \( j \geq 2 \)

\[
(h/d_j)^{-s}d_j^{d/2}(|\nabla (g - \psi)|_{L_2(\Omega_j)} + d_j^{-1}||g - \psi||_{L_2(\Omega_j)})
\]

\[
\lesssim (h/d_j)^{-s}d_j^{d/2}(|\nabla (g - \psi)|_{L_\infty(\Omega_j)} + d_j^{-1}||g - \psi||_{L_\infty(\Omega_j)})
\]

\[
\lesssim (h/d_j)^{-s}d_j^{d/2}h^r|g|_{W^{2,s+1}(\Omega_j')}
\]

\[
\lesssim (h/d_j)^{-s}d_j^{d/2}h^{2-d-r-1}
\]

\[
= h(h/d_j)^{r-1-s} = h(h/2)^{r-1-s} \lesssim h2^{-j(r-1-s)}.
\]

For \( j = 0, 1 \), we have \( d_j \approx h \). Using standard \( H^2 \) regularity \( \|g\|_{H^2(\Omega)} \lesssim \|\delta^{x_0}\|_{L_2(\Omega)} \lesssim h^{-d/2} \), we thus have

\[
(h/d_j)^{-s}d_j^{d/2}(|\nabla (g - \psi)|_{L_2(\Omega_j)} + d_j^{-1}||g - \psi||_{L_2(\Omega_j)}) \lesssim h^{d/2} h \|g\|_{H^2(\Omega)} \lesssim h.
\]

Inserting the last two inequalities into (4.21) yields

\[
\|\sigma_{x_0} g - g_h\|_{L_1(\Omega)} + \|\sigma_{x_0} \nabla (g - g_h)\|_{L_1(\Omega)} \lesssim h^{d+r-1}
\]

\[
\lesssim h + h \sum_{j=2}^J 2^{-j(r-1-s)} + \sum_{j=0}^J (h/d_j)^s d_j^{d/2-1} \|g - g_h\|_{L_2(\Omega_j')}
\]

\[
\lesssim h(1 + (\ln h)^{d+r-1}) + \sum_{j=0}^J (h/d_j)^s d_j^{d/2-1} \|g - g_h\|_{L_2(\Omega_j')}.
\]

Here we have used the fact that when \( r - 1 - s > 0 \), a geometric sum yields \( \sum_{j=2}^J 2^{-j(r-1-s)} \lesssim 1 \), and when \( r - 1 - s = 0 \) we have \( \sum_{j=2}^J 2^{-j(r-1-s)} = \sum_{j=2}^J 1 \approx J \approx \ln h \).

**Step 4: Second duality argument.** We now approach the term \( \sum_{j=0}^J (h/d_j)^s d_j^{d/2-1} \|g - g_h\|_{L_2(\Omega_j')} \). This is the heart of the proof. We first clean up our notation a little by noting

\[
\sum_{j=0}^J (h/d_j)^{-s}d_j^{d/2-1} \|g - g_h\|_{L_2(\Omega_j')} \lesssim \sum_{j=0}^J (h/d_j)^{-s}d_j^{d/2-1} \|g - g_h\|_{L_2(\Omega_j')}.
\]

For any \( j \), we have

\[
\|g - g_h\|_{L_2(\Omega_j)} = \sup_{v_j \in C_0^\infty(\Omega_j), \|v_j\|_{L_2(\Omega)} = 1} (g - g_h, v_j).
\]
Given such a $v_j$, we let $z_j$ solve $L(z_j, w) = (v_j, w)$, $w \in H^1(\Omega)$. Then for $\chi \in S_h$ and properly chosen $v_j$ and associated $z_j$,

\[
\|g - g_h\|_{L^2(\Omega')} \lesssim L(z_j, g - g_h) = L(z_j - \chi, g - g_h) \\
\leq \|z_j - \chi\|_{W^{m,1}(\Omega(\Omega')')}\|g - g_h\|_{W^{1,1}(\Omega(\Omega')')} + \|z_j - \chi\|_{H^1(\Omega')}\|g - g_h\|_{H^1(\Omega')}.
\]

(4.29)

We first compute for properly chosen $\chi \in S_h$

\[
\|z_j - \chi\|_{H^1(\Omega')} \lesssim h\|z_j\|_{H^2(\Omega)} \lesssim h\|v_j\|_{L^2(\Omega)} \lesssim h.
\]

(4.30)

Also,

\[
\|z_j - \chi\|_{W^{m,1}(\Omega(\Omega')')} \lesssim h^r|z_j|_{W^{\phi+1}(\Omega(\Omega')').
\]

(4.31)

We compute for $y \in \Omega \setminus \Omega_j$ and $|\alpha| = r + 1$ that

\[
D^\alpha z_j(y) = D^\alpha v_j, G^\alpha = (v_j, D^\alpha G^\alpha) \leq \|v_j\|_{L^2(\Omega_j)}d_j^{d/2}\|D^\alpha G^\alpha\|_{L^\infty(\Omega_j)} \\
\leq d_j^{d/2}d_j^{2-d-|\alpha|} = d_j^{1-r-d/2},
\]

since $|y - \bar{x}| \geq d_j$ for any $y \in \Omega \setminus \Omega_j$, $\bar{x} \in \Omega_j$. Finally, we may compute that

\[
\|g - g_h\|_{L^2(\Omega)} \leq h \inf_{\chi \in S_h} \|g - \chi\|_{H^1(\Omega)} \lesssim h^2\|g\|_{H^2(\Omega)} \lesssim h^{2-d/2}.
\]

(4.33)

Inserting the previous four inequalities into (4.29) and the result into (4.27) yields

\[
\sum_{j=0}^J (h/d_j)^{s}d_j^{d/2-1}\|g - g_h\|_{L^2(\Omega')}
\]

\[
\lesssim h^{d/2-1}\|g - g_h\|_{L^2(\Omega')} + \sum_{j=0}^J (h/d_j)^{s}d_j^{d/2-1}(h\|g - g_h\|_{H^1(\Omega')} + h^r d_j^{1-r-d/2}\|g - g_h\|_{W^1(\Omega)})
\]

\[
\lesssim h + \sum_{j=0}^J (h/d_j)^{s+1}d_j^{d/2}\|g - g_h\|_{H^1(\Omega_j)} + (h/d_j)^{r-s}\|g - g_h\|_{W^{1,1}(\Omega)}.
\]

(4.34)

\textbf{Step 5: Double kickback.} Recall now that $h/d_j \leq \frac{1}{M}$, $j \geq 0$. We now insert (4.34) into (4.21) while recalling the results of (4.26) to find that

\[
\|\sigma_{x_0, h}^s(g - g_h)\|_{L^1(\Omega)} + \|\sigma_{x_0, h}^{-s} \nabla (g - g_h)\|_{L^1(\Omega)}
\]

\[
\leq \sum_{j=0}^J (h/d_j)^{s}d_j^{d/2}\|g - g_h\|_{H^1(\Omega_j)}
\]

\[
\lesssim h(1 + (\frac{1}{M})^{d-s-1}) + \frac{1}{M} \sum_{j=0}^J (h/d_j)^{s}d_j^{d/2}\|g - g_h\|_{H^1(\Omega_j)} + \sum_{j=0}^J (h/d_j)^{r-s}\|g - g_h\|_{W^{1,1}(\Omega)}.
\]

(4.35)
Taking $M$ large enough to kick back the terms involving $\|g - g_h\|_{L^1(\Omega)}$ above yields
\[
\|\sigma_{x_0,h}^s (g - g_h)\|_{L^1(\Omega)} + \|\sigma_{x_0,h}^{-s} \nabla (g - g_h)\|_{L^1(\Omega)} \\
\lesssim h(1 + (\ln \frac{1}{h})^{\delta_{x,\cdot} - 1}) + \sum_{j=0}^J (h/d_j)^{r-s} \|g - g_h\|_{W^{1,1}(\Omega)}. 
\] (4.36)

But $\sum_{j=0}^J (h/d_j)^{r-s} = \sum_{j=0}^J (h/2^j M)^{r-s} = M^{s-r} \sum_{j=0}^J 2^{-j(r-s)} \lesssim M^{-1}$, since $r - s \geq 1$. In addition, because $\sigma_{x_0,h}^{-1} \geq 1$ we have $\|g - g_h\|_{W^{1,1}(\Omega)} \leq \|\sigma_{x_0,h}^s (g - g_h)\|_{L^1(\Omega)} + \|\sigma_{x_0,h}^{-s} \nabla (g - g_h)\|_{L^1(\Omega)}$. We may thus take $M$ large enough to reabsorb the term involving $\|g - g_h\|_{W^{1,1}(\Omega)}$ above, which completes the proof.

\[\square\]

4.3 $W^{\infty,1}$ estimates on polyhedral domains

In this section we briefly discuss issues that arise when trying to prove pointwise error estimates on polygonal or polyhedral domains. In addition, we shall also discuss attempts to prove such error estimates on solution-adapted meshes.

4.3.1 Statement of results

We state there a theorem from [38]; cf. [43, 15] for other results (including the 2D case).

**Theorem 4.3.1** Assume that $u \in H^1_0(\Omega)$ weakly satisfies $-\Delta u = f$ on a convex polyhedral domain $\Omega \subset \mathbb{R}^3$ with $f$ sufficiently smooth. Let also $u_h \in S_h$ be its finite element approximation, with $S_h \subset \mathcal{S}_h(\Omega)$ a Lagrange finite element space on a quasi-uniform mesh. Then
\[
\|u_h\|_{W^{\infty,1}(\Omega)} \lesssim \|u\|_{W^{\infty,1}(\Omega)}, 
\] (4.37)
and consequently
\[
\|u - u_h\|_{W^{\infty,1}(\Omega)} \lesssim \inf_{\chi \in S_h} \|u - \chi\|_{W^{1,1}(\Omega)}. 
\] (4.38)

The proof of the above theorem is very similar to the one given in the previous subsection for bounding $|(u - u_h)(x_0)|$. One first represents the error by using regularized Green’s and $\delta$ functions. Let $\delta^{x_0}$ be a discrete $\delta$ function as above, and let $\nu$ be a unit vector in $\mathbb{R}^3$. Then for $\chi \in S_h$,
\[
\nabla \chi \cdot \nu = (\nabla \chi, \delta^{x_0} \nu) = -(\chi, \nabla \delta^{x_0} \cdot \nu). 
\] (4.39)

Our regularized Green’s function $g \in H^1_0(\Omega)$ then solves $-\Delta g = -\nabla \delta^{x_0} \cdot \nu$. We may thus represent the error and proceed much as in the previous subsection, using dyadic decompositions, local energy estimates, a double kickback argument, etc. The main difference comes in the regularity of the Green’s function. Because $\partial \Omega$ is no longer smooth, (4.22) giving the decay of the Green’s function near the singularity only holds for very restricted $\alpha, \beta$. In fact, if we consider only integer orders we must have $|\alpha| \leq 1$ and $|\beta| \leq 1$, which is not sufficient. The main advance in [38] was to prove...
Hölder estimates for Green’s functions on convex polyhedral domains. More precisely, the estimates needed are of the form

\[
\frac{\lvert \partial_x \partial_{\xi_j} G(x, \xi) - \partial_y \partial_{\xi_j} G(y, \xi) \rvert}{|x - y|^{\sigma}} \lesssim |x - \xi|^{-3-\sigma} + |y - \xi|^{-3-\sigma}.
\]

It is shown in [38] that (4.40) holds on any convex polyhedral domain (under appropriate assumptions) for \(0 < \sigma < 1\) depending on the maximum interior opening angle of \(\Omega\). Recall that \(\nabla u\) itself is not generally bounded on nonconvex domains; in particular there will in this case be edge singularities of the form \(\rho^{\pi/\omega}\) with \(\omega > \pi\) and thus \(\nabla u \sim \rho^{\pi/\omega - 1}\) has negative exponent and blows up near the edge \(e\). Thus the restriction that \(\Omega\) is convex is quite natural here.

As an interesting historical note, stability of the FEM in \(W^{1,1}\) on 2D convex polygonal domains was proved in [43]. The text [15] (and earlier versions) contains a 3D version of the result as well, but with assumptions that restrict the maximum interior edge opening angle to be strictly less than \(2\pi/3\). This restriction is discussed more explicitly in [35], where similar bounds are proved for the Stokes problem. Instead of using Green’s function Hölder bounds as above, they employed \(W^{p,2}\) regularity results which were more readily available in the literature at the time. With regard to the restriction that the edge opening angles be less than \(2\pi/3\), the authors stated that the restriction is natural because of the relationship between the angle condition and \(W^{p,2}\) regularity needed to obtain pointwise bounds on both the continuous and discrete solution. Such \(W^{p,2}\) estimates do not hold for sufficiently large \(p\) on arbitrary convex polyhedral domains, while the Green’s function estimates (4.40) do. The takeaway lesson is that developing multiple ways to understand PDE regularity is essential. Here both \(W^{p,2}\) and Hölder regularity come into play and give meaningful information, but gaining a sharp understanding of pointwise behavior relies on the latter type of estimates.

### 4.3.2 Mesh regularity

Recall Céa’s Lemma

\[
\|u - u_h\|_{H^1(\Omega)} \lesssim \inf_{\chi \in S_h} \|u - \chi\|_{H^1(\Omega)},
\]

which holds whenever the associated bilinear form is continuous and coercive. This stability estimate holds for any mesh, and in particular on the shape-regular but not quasiuniform meshes generated by adaptive refinement procedures. The \(W^{1,1}\) stability result (4.37) and almost-best-approximation result (4.38) look similar, but their proof in [38] assumes quasiuniform meshes. It is natural to ask whether such estimates also hold on shape-regular (adaptive) grids. Several papers have tried to answer this question, but it remains at least partially open.

We briefly describe the approach of Eriksson [28]. To understand his assumption, assume that \(T\) is a shape regular grid. We construct a mesh size function \(h_T \in W^{1,1}_0(\Omega)\) as follows. For each \(T \in \mathcal{T}\), let \(h_T = \text{diam}(T)\) as usual. For each vertex \(z\) in \(T\), we then let \(h_T(z)\) be the average of the meshes sizes \(h_T\) over all elements \(T\) sharing the vertex \(z\). Finally, \(h_T\) is taken to be continuous and piecewise linear, so specifying it at vertices determines it uniquely. The shape regularity of \(T\) guarantees that \(h_T(x) \approx h_T\) whenever \(x \in T \in \mathcal{T}\), and in addition that

\[
|\nabla h_T| \lesssim 1
\]

with constant depending on the shape regularity of \(\Omega\). These facts are easy consequences of the observation that for shape-regular grids, neighboring elements (sharing a face) always have comparable diameters, and the number of elements sharing a given vertex is always uniformly bounded.
Eriksson’s idea was to assume instead of (4.41) that

\[ |\nabla h_T| \lesssim \mu, \quad \mu \text{ sufficiently small.} \tag{4.42} \]

This condition is significantly more restrictive than shape regularity, but still allows for meaningful mesh grading. For example, “smooth” mesh gradings which both satisfy (4.42) and which optimally resolve corner singularities may be constructed. Eriksson proved $L_\infty$ estimates on 2D polygonal domains assuming a condition much like (4.42), while [24] contains a proof of the $W^{\infty,1}$ stability estimate (4.37) on convex polygonal and polyhedral domains.