1 10.1: Sequences

Definitions:

sequence $\{a_n\}$:

$\lim_{n\to\infty} a_n = L$

Sequences defined using real-valued functions:

Limit Laws: Given a_n and b_n converge and c is a constant, then:

1. $\lim_{n\to\infty} (a_n + b_n) =$
2. $\lim_{n\to\infty} (a_n - b_n) =$
3. $\lim_{n\to\infty} ca_n =$
4. $\lim_{n\to\infty} a_n b_n =$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} =$
6. If $\lim_{n\to\infty} |a_n| = 0$, then
7. If f is a continuous function and $a_n \to L$, then
8. **Squeeze Theorem**:

Examples: Find the limits of the following sequences:

$a_n = \frac{\ln(n + e^{3n})}{n}$

$$a_n = \left(1 + \frac{3}{n}\right)^{n/2}$$

$$a_n = \arctan\left(\frac{n}{n+1}\right)$$

$$a_n = \frac{(-1)^{n+1}}{2n+1}$$

More Definitions:

Monotonic sequence:

$\{a_n\}$ is increasing for $n \geq N$ if and only if (Implications:)

1. $a_{n+1} - a_n > 0$;
2. $\dfrac{a_{n+1}}{a_n} > 1$ if $a_n > 0$;
3. If $a_n = f(n)$ for some real-valued function f, then $f' > 0$.

$\{a_n\}$ is decreasing for $n \geq N$ if and only if (Implications:)

$\{a_n\}$ is bounded above (below) if and only if

Monotone Sequence Theorem:

Determine if the sequence $a_n = \dfrac{\ln n}{n}$ is monotonic and bounded.

Given the sequence defined recursively by $a_1 = 1$, $a_{n+1} = \sqrt{3 + a_n}$ is increasing and bounded above by 3, find the limit.

On Beyond Average:

Find the limit of $a_n = (\sqrt{n+1} - \sqrt{n})\sqrt{n + \frac{1}{2}}$

Given $a_n = \frac{1000^n}{n!}$, show a_n is decreasing (for $n >$some N) and bounded below. What is the limit of this sequence, and why?

Proof by Induction (Appendix E)

Example:

Given $a_0 = 1$ and $a_{n+1} = \frac{1}{3}a_n + 1$:

a) Show a_n is increasing and $0 < a_n < 2$ for all n.

b) Find $\lim_{n\to\infty} a_n$.