1 10.2: Series

Definitions:

Infinite Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_{n-1} + a_n + \ldots$

a_n the <u>terms</u> of the series

Nth Partial Sum: $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_{N-1} + a_N$

Convergent Series: $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence S_N of partial sums converges. If $S_N \to S$, we say $\sum_{n=1}^{\infty} a_n$ converges to S

Divergent Series: if S_N diverges, then we say the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Special (Summable) Kinds of Series:

Geometric Series: a series of the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$

r = ratio (multiply each term by to get the next term)

a: starting value

Find the values of r for which $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent and find the sum.

$$S_N = \sum_{n=1}^{N} ar^{n-1} = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + \cancel{ar^{N-1}}$$

$$-rS_N = -\cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{ar^4} + \dots + \cancel{ar^{N-1}} + ar^N$$

$$S_N - rS_N = a - ar^N$$

$$S_N(1-r) = a - ar^N$$

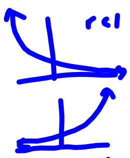
$$\boxed{S_N = \frac{a - ar^N}{1-r}}$$

What happens as $N \to \infty$?

if $|r|<1$, $ar^N \to 0$

if $|r|>1$, ar^N diverges

if $r=1$ or -1 $\sum ar^{n-1}$ divergent



$\therefore \sum_{n=1}^{\infty} ar^{n-1}$ is convergent if and only if $|r|<1$ and converges to $\boxed{\frac{a}{1-r}}$

Examples:

Evaluate $\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n$ $= 1 - \frac{4}{5} + \frac{16}{25} - \frac{64}{125} + \ldots$

Geometric

$a = 1$

$r = -\frac{4}{5}$ $\quad |r| < 1$

Series converges to $\frac{a}{1-r} = \frac{1}{1-\left(-\frac{4}{5}\right)} = \frac{1}{\frac{9}{5}} = \boxed{\frac{5}{9}}$

Write $0.\overline{27}$ as
(a) an infinite series
(b) a (reduced) fraction

$$0.\overline{27} = .\underline{27}\,\underline{27}\,\underline{27}\,\underline{27}\,\underline{27}\ldots$$

$$= \frac{27}{100} + \frac{27}{10^4} + \frac{27}{10^6} + \frac{27}{10^8} + \frac{27}{10^{10}} + \ldots$$

(a) $$= \boxed{\sum_{n=1}^{\infty} \frac{27}{10^{2n}}} = \boxed{\sum_{n=1}^{\infty} \frac{27}{100^n}} = \sum_{n=1}^{\infty} \left(\frac{27}{100}\right)\left(\frac{1}{100}\right)^{n-1}$$

(b) Geometric $a = \frac{27}{100}$

$r = \frac{1}{100}$ $|r| < 1$

$$\therefore \text{ converges to } \frac{a}{1-r} = \frac{\left(27/100\right)100}{\left(1 - 1/100\right)100} = \frac{27}{100-1} = \frac{27}{99} = \boxed{\frac{3}{11}}$$

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Telescoping Series:

Find the sum of $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+2}\right) = \left(1-\cancel{\tfrac{1}{3}}\right)+\left(\tfrac{1}{2}-\cancel{\tfrac{1}{4}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}\right)+\left(\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{6}}\right)+\dots+\left(\cancel{\tfrac{1}{N-2}}-\cancel{\tfrac{1}{N}}\right)+\left(\cancel{\tfrac{1}{N-1}}-\tfrac{1}{N+1}\right)+\left(\cancel{\tfrac{1}{N}}-\tfrac{1}{N+2}\right)$$

$$S_N = 1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}$$

As $N\to\infty$ $\quad S_N \to 1+\frac{1}{2} = \frac{3}{2}$

Find the sum of $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$. $= \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \ldots$ NOT Geometric

Partial Fractions:

$$\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$2 = A(n+2) + Bn$$

if $n=0$ $\quad 2 = 2A \rightarrow A = 1$

if $n=-2$ $\quad 2 = -2B \rightarrow B = -1$

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$ previous example

$\therefore \sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ converges to $\boxed{\frac{3}{2}}$

Properties of Convergent Series:

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, then:

i) $\sum_{n=0}^{\infty}(a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$

ii) $\sum_{n=0}^{\infty}(a_n - b_n) = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} b_n$

iii) $\sum_{n=0}^{\infty} c a_n = c \sum_{n=0}^{\infty} a_n$

$$\left(\sum_{n=0}^{\infty} \frac{5+2^n}{6^n} = \sum_{n=0}^{\infty} \frac{5}{6^n} + \sum_{n=0}^{\infty} \frac{2^n}{6^n} \right)$$

$\frac{2^n}{6^n} = \left(\frac{1}{3}\right)^n$

Both convergent geometric series

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Tests for Convergence of Series: (continued through 10.4)

I. The Test for Divergence (or **Divergence Test**): If $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$

Contrapositive If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent

$$\sum_{n=0}^{\infty} \frac{n+3}{2n+1}$$

$a_n = \frac{n+3}{2n+1} \longrightarrow \frac{1}{2}$

$\therefore$ $\boxed{\sum_{n=0}^{\infty} \frac{n+3}{2n+1} \text{ is divergent by the Test for Divergence}}$

The Nth partial sum of a series is given by $s_N = \dfrac{N+1}{2N+3}$. Find the sum of the series or explain why the series is divergent.

$$S_N = \frac{N+1}{2N+3} \to \frac{1}{2}$$

$\therefore \sum_{n=1}^{\infty} a_n$ is convergent and converges to $\frac{1}{2}$.

Claim

$$a_n = S_n - S_{n-1}$$

$$= \frac{n+1}{2n+3} - \frac{(n-1)+1}{2(n-1)+3}$$

$$= \frac{n+1}{2n+3}\frac{(2n+1)}{(2n+1)} - \frac{n}{2n+1}\frac{(2n+3)}{(2n+3)}$$

$$= \frac{\cancel{2n^2} + \cancel{3n} + 1 - (\cancel{2n^2} + \cancel{3n})}{(2n+1)(2n+3)}$$

$$\boxed{a_n = \frac{1}{(2n+1)(2n+3)}}$$

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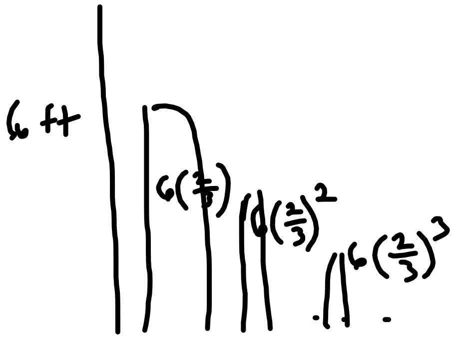
A ball bounces back to $\frac{2}{3}$ of its original height. If the ball is dropped from a height of 6 feet and allowed to bounce forever, what is the total distance the ball will travel?



$$D = 6 + \underbrace{12\left(\tfrac{2}{3}\right) + 12\left(\tfrac{2}{3}\right)^2 + 12\left(\tfrac{2}{3}\right)^3 + \dots}_{\text{Geometric Series}}$$

$$= 6 + \sum_{n=0}^{\infty} 12\left(\tfrac{2}{3}\right)^n$$

$a = 12\left(\frac{2}{3}\right) = 8$

$r = \frac{2}{3}$

$\frac{a}{1-r}$

$$= 6 + \frac{8 \cdot 3}{\left(1 - \frac{2}{3}\right) \cdot 3}$$

$$= 6 + \frac{24}{3-2}$$

$$= \boxed{30 \text{ feet}}$$

Harmonic Series

Prove the converse of the Test for Divergence is false by showing that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (even though $\frac{1}{n} \to 0$).

Test for Divergence: If $\sum a_n$ converges, then $a_n \to 0$

Converse: If $a_n \to 0$, then $\sum a_n$ converges ✗ FALSE

Counter Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad a_n = \frac{1}{n} \to 0$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots + \frac{1}{16} + \frac{1}{17} + \ldots$$

$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \underbrace{\frac{1}{16} + \ldots + \frac{1}{16}} + \underbrace{\frac{1}{32} + \ldots}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

each time number of terms double, add $\frac{1}{2}$ to sum.

Can continue this indefinitely

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent