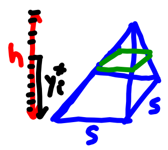
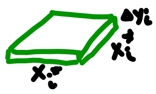
1 7.2: Volume by Slicing

The volume of a prism with a base of area B and a height h is given by $V = Bh$

If the height is not constant, we can, in many cases, "slice" the solid thin enough so that the height is constant in a given slice

Example: Find the volume of a square pyramid whose height is h and whose base is s by s.





$$V_{slice} = (x_i^*)^2 \Delta y_i$$

$$V_{total} = \lim_{\|P\| \to 0} \sum_{i=1}^{n} (x_i^*)^2 \Delta y_i$$

$$= \int x^2 \, dy$$

$$= \int_0^h \left(\frac{s}{h}(h-y)\right)^2 dy$$

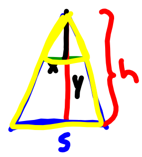
$$= \frac{s^2}{h^2} \int_0^h (h^2 - 2hy + y^2) \, dy$$

$$= \frac{s^2}{h^2} \left(h^2 y - \cancel{2}h\left(\frac{1}{\cancel{2}}y^2\right) + \frac{1}{3}y^3 \right|_0^h$$

$$= \frac{s^2}{\cancel{h^2}} \left(\cancel{h^3 - h^3} + \frac{1}{3}h^{\cancel{3}} \right)$$

$$= \boxed{\frac{1}{3}s^2 h}$$

relationship between x and y



$$\frac{x}{s} = \frac{h-y}{h}$$

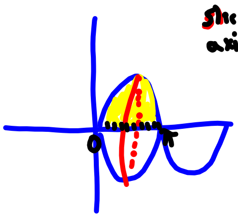
$$x = \frac{s}{h}(h-y)$$

If the solid is generated by rotating a region about an axis, then the bases of the slices will be circles and the area will be $B = \pi r^2$. If there is a hole in the solid, we can find the volume by finding the volume of the outer solid (without the hole) minus the volume the hollowed out portion.

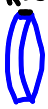
Examples:

Find the volume of the solid formed by rotating the region above the x-axis (closest to the origin) bounded by the curves $y = \sin x$ and $y = 0$ about the x-axis.

slice ⊥ to axis of rotation



h=dx



$V_{slice} = \pi r^2 h$

$r = y$

$h = dx$

$$V = \int \pi y^2 \, dx$$

$$= \int_0^{\pi} \pi (\sin x)^2 \, dx$$

$$= \pi \int_0^{\pi} \sin^2 x \, dx \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$$= \pi \int_0^{\pi} \tfrac{1}{2}(1 - \cos(2x)) \, dx$$

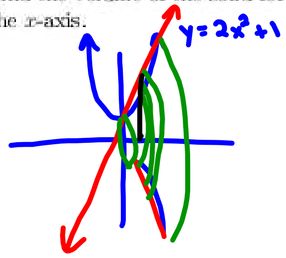
Let $u = 2x$

$$= \frac{\pi}{2}\left(x - \tfrac{1}{2}\sin 2x \,\Big|_0^{\pi}\right.$$

$$= \frac{\pi}{2}\left(\pi - \tfrac{1}{2}\sin 2\pi - 0 + 0\right)$$

$$= \boxed{\frac{\pi^2}{2}}$$

Find the volume of the solid formed by rotating the region bounded by $y = 2x^2 + 1$ and $y = 3x$ about the x-axis.



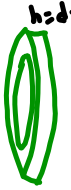
Intersection: $2x^2+1 = 3x$

$2x^2 - 3x + 1 = 0$

$(2x-1)(x-1) = 0$

$x = \frac{1}{2} \quad x = 1$

h=dx



$V = \pi r_{out}^2 h - \pi r_{in}^2 h$

$r_{out} = y_{out} = 3x$

$r_{in} = y_{in} = 2x^2+1$

$h = dx$

$$V = \int_{\frac{1}{2}}^{1} \left(\pi(3x)^2 - \pi(2x^2+1)^2\right) dx$$

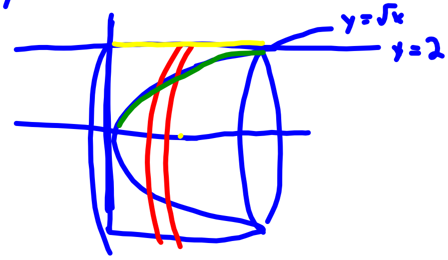
$9x^2 \qquad 4x^4+4x^2+1$

$$= \pi \int_{\frac{1}{2}}^{1} \left((3x)^2 - (2x^2+1)^2\right) dx$$

$$= \pi \left(3x^3 - \frac{4}{5}x^5 - \frac{4}{3}x^3 - x \right|_{\frac{1}{2}}^{1}$$

$$= \boxed{\pi\left[\left(3 - \frac{4}{5} - \frac{4}{3} - 1\right) - \left(3\left(\frac{1}{2}\right)^3 - \frac{4}{5}\left(\frac{1}{2}\right)^5 - \frac{4}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{2}\right)\right]}$$

Find the volume of the solid formed by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 0$, and $y = 2$ about the x-axis.

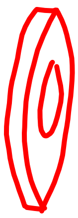


Intersection:

$\sqrt{x} = 2$

$x = 4$

$V = \pi r_{out}^2 h - \pi r_{in}^2 h$



$r_{out} = y_{out} = 2$

$r_{in} = y_{in} = \sqrt{x}$

$h = dx$

$$V = \int_0^4 \left(\pi(2)^2 - \pi(\sqrt{x})^2\right) dx$$

$$= \pi \int_0^4 (4 - x)\, dx$$

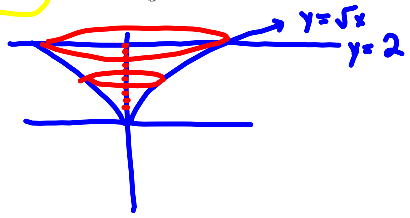
$$= \pi \left(4x - \tfrac{1}{2}x^2\right)\Big|_0^4$$

$$= \pi(16 - 8) = \boxed{8\pi}$$

Find the volume of the solid formed by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 0$, and $y = 2$ about the y-axis.

$x = y^2$

"Intersection:
$y^2 = 0$
$y = 0$"



$V = \pi r^2 h$

$r = x = y^2$

$h = dy$

$$V = \int_0^2 \pi (y^2)^2 \, dy$$
$$= \pi \int_0^2 y^4 \, dy$$
$$= \pi \left. \tfrac{1}{5} y^5 \right|_0^2$$
$$= \boxed{\frac{32\pi}{5}}$$

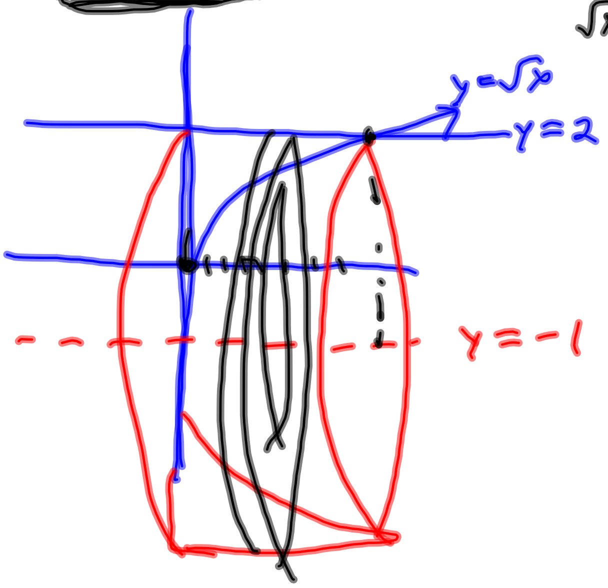
Sep 2-11:31 AM

On Beyond Average:

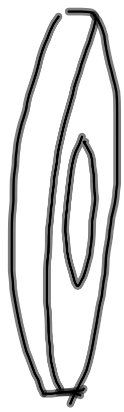
Find the volume of the solid formed by rotating the region in the previous example about the line $y = -1$.

~~Find the volume of the solid formed by rotating~~ the region bounded by the curves $y = \sqrt{x}$, $x = 0$, and $y = 2$ ~~about the y-axis~~.

$\sqrt{x} = 2 \Rightarrow x = 4$



$V_{outer} - V_{inner}$



$h = dx$

$r_{outer} = 2 - (-1) = 3$

$r_{inner} = \sqrt{x} - (-1)$

$= \sqrt{x} + 1$

$$V = \int_0^4 \pi (3)^2 \, dx - \int_0^4 \pi (\sqrt{x}+1)^2 \, dx$$

$$= \pi \int_0^4 \left((9) - (x + 2\overset{x^{1/2}}{\sqrt{x}} + 1) \right) dx$$

$$= \pi \left(9x - \tfrac{1}{2}x^2 - 2 \cdot \tfrac{2}{3} x^{3/2} - x \right|_0^4$$

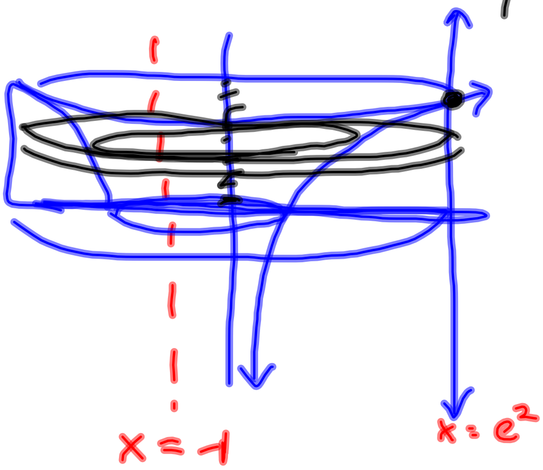
$$= \boxed{\pi \left(36 - 8 - \tfrac{4}{3} \cdot 4^{3/2} - 4 \right)} \qquad (2\sqrt{4})^3$$

$$= \pi \left(24 - \tfrac{4}{3} \cdot 8 \right)$$

$$= \pi \left(\tfrac{40}{3} \right) = \boxed{\frac{40\pi}{3}}$$

Set up, but do not evaluate, an integral to find the volume of the solid formed by rotating the region bounded by $y = \ln x$, $y = 0$, and $x = e^2$ about the line $x = -1$.

$x = e^y$ boundary

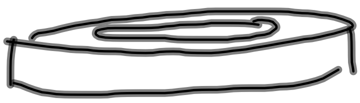


$e^y = e^2 \Rightarrow y = 2$

$V_{outer} \cdot V_{inner}$

$h = dy$

$r_{outer} = e^2 - (-1) = e^2 + 1$

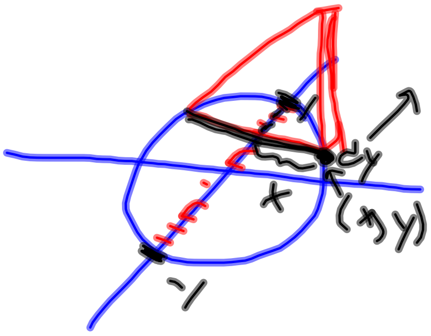


$r_{inner} = e^y - (-1)$

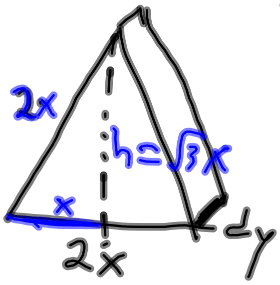
$= e^y + 1$

$$\boxed{V = \int_0^2 \pi (e^2+1)^2 \, dy - \int_0^2 \pi (e^y+1)^2 \, dy}$$

The base of a solid is the unit circle in the x-y plane. Cross-sections perpendicular to the ~~base~~ y-axis are equilateral triangles. Find the volume of the solid.



equation of circle $\Rightarrow x^2+y^2=1$



$$V_{slice} = \frac{1}{2}(2x)(\sqrt{3}\,x)\,dy$$
$$= \sqrt{3}\,x^2\,dy$$

Pythagoras

$$h = \sqrt{(2x)^2 - x^2}$$
$$= \sqrt{4x^2 - x^2}$$
$$= \sqrt{3x^2}$$
$$= \sqrt{3}\,x$$

$$V_{tot} = \int \sqrt{3}\,x^2\,dy \qquad x^2 = 1 - y^2$$

$$V = \int_{-1}^{1} \sqrt{3}(1-y^2)\,dy$$

symmetry

$$= 2\int_0^1 \sqrt{3}(1-y^2)\,dy$$
$$= 2\sqrt{3}\left(y - \frac{1}{3}y^3\right)\Big|_0^1$$
$$= \boxed{2\sqrt{3}\left(1 - \frac{1}{3}\right)}$$
$$= \boxed{\frac{4\sqrt{3}}{3}}$$