1 8.9: Improper Integrals

Improper integrals are integrals involving a region which is unbounded in some sense. There are two types of improper integrals among the examples that follow, each of which is handled similarly:

Cannot apply FTC on this problem

$\int_a^b f(x)\,dx = F(b) - F(a)$

Examples: $\int_0^\infty e^{-3x}\,dx = \lim_{a\to\infty} \int_0^a e^{-3x}\,dx$

$u = -3x$

$du = -3dx$

$= \lim_{a\to\infty} -\frac{1}{3} e^{-3x} \Big|_0^a$

$= \lim_{a\to\infty} -\frac{1}{3} \underbrace{e^{-3a}}_{0} + \frac{1}{3} e^0$

$= \boxed{\frac{1}{3}}$. The Integral converges to $\frac{1}{3}$

$$\int_0^\infty xe^{-3x}\,dx = {}^{*}\lim_{a\to\infty} \int_0^a xe^{-3x}\,dx^{*}$$

IBP Tabular

$$\begin{array}{ll} x & e^{-3x} \\ 1 & -\frac{1}{3}e^{-3x} \\ 0 & \frac{1}{9}e^{-3x} \end{array}$$

$$= \lim_{a\to\infty} -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\Big|_0^a$$

$$= \lim_{a\to\infty} -\frac{1}{3}ae^{-3a} - \frac{1}{9}e^{-3a} + 0 + \frac{1}{9}$$

L'Hospital's Rule

$$-\frac{1}{3}\lim_{a\to\infty}\frac{a}{e^{3a}} = -\frac{1}{3}\lim_{a\to\infty}\frac{1}{3e^{3a}} \to 0$$

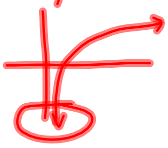
$$= \boxed{\frac{1}{9}}$$ converges to $\frac{1}{9}$

Cannot use FTC since $\frac{1}{x-4} \to \overset{\text{unbounded}}{\infty}$ as $x \to 4$

$$\int_4^6 \frac{1}{x-4}\,dx = \lim_{a\to 4^+}\int_a^6 \frac{1}{x-4}\,dx$$

$$= \lim_{a\to 4^+} \ln|x-4| \Big|_a^6$$

$$= \lim_{a\to 4^+} \ln 2 - \ln\underset{-\infty}{(a-4)}$$



$\therefore$ $\boxed{\text{The integral diverges to } \infty}$

$$\int_3^6 \frac{1}{x-4}\,dx = \cancel{\ln|x-4|\Big|_3^6 = \ln 2 - \ln 1 = \boxed{\ln 2}}$$

$\frac{1}{x-4}$ is unbounded at $x=4$, so cannot apply FTC

$$= \int_3^4 \frac{1}{x-4}\,dx + \underbrace{\int_4^6 \frac{1}{x-4}\,dx}_{\text{diverges}}$$

$$\therefore \boxed{\int_3^6 \frac{1}{x-4}\,dx \text{ diverges}}$$

Find all values of p for which $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

$$= \lim_{a\to\infty} \int_1^a x^{-p+1}\,dx$$

$$= \lim_{a\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^a$$

$$= \lim_{a\to\infty} \frac{1}{1-p}\left(a^{1-p} - 1\right)$$

We want a finite answer, ("a" in the denominator) which happens when exponent is negative

$1-p < 0$

$-p < -1$

$p > 1$

if $p>1$, $a^{1-p} \to 0$, so the integral converges to $\frac{-1}{1-p} = \frac{1}{p-1}$

if $p<1$, $a^{1-p} \to \infty$, so the integral diverges to ∞

if $p=1$ $\lim_{a\to\infty}\int_1^a x^{-1}\,dx = \lim_{a\to\infty} \ln x\Big|_1^a = \lim_{a\to\infty} \underbrace{\ln a}_{\infty} - \underbrace{\ln 1}_{0}$ diverges

$\therefore$ $\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ converges if and only if } p>1}$

NOTE $\int_0^1 \frac{1}{x^p}\,dx$ converges if and only if $p<1$

Comparison Theorem: Suppose f and g are continuous and $0 \leq f(x) \leq g(x)$ for $x > N$:

a) If $\int_N^\infty g(x)\,dx$ converges, then $\int_N^\infty f(x)\,dx$ converges

finite area

b) If $\int_N^\infty f(x)\,dx$ diverges, then $\int_N^\infty g(x)\,dx$ diverges

infinite area

NOTE: if $\int_N^\infty g(x)\,dx$ diverges or $\int_N^\infty f(x)\,dx$ converges, we know NOTHING

On Beyond Average:

$$\int_3^\infty \frac{x+1}{x^2-4}\,dx =$$

As $x \to \infty$, $\frac{x+1}{x^2-4}$ behaves like $\frac{x}{x^2} = \frac{1}{x}$

$p=1$ $\quad \int_3^\infty \frac{1}{x^1}\,dx$ diverges

we want $f(x) = \frac{1}{x}$

Is $\frac{1}{x} < \frac{x+1}{x^2-4}$?

$\cancel{x^2} - 4 < \cancel{x^2} + x$

$x > -4$ ✓

Comparison Theorem: Suppose f and g are continuous and $0 \leq f(x) \leq g(x)$ for $x > N$:

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$\therefore \int_3^\infty \frac{x+1}{x^2-4}\,dx$ diverges by Comparison with $\int_3^\infty \frac{1}{x}\,dx$

$$\int_1^{\infty} \frac{1}{x(x^2+1)}\,dx =$$ Glance: behaves like $\int_1^{\infty} \frac{1}{x^3}dx$ converge $3>1$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

$x=0$ $\quad 1 = A$

$Bx^2 + Cx$ $\quad 1 = \underline{x^2} + 1 + \underline{Bx^2} + \underline{Cx}$

x^2 $\quad 0 = 1 + B \to B = -1$

x $\quad 0 = C$

$$= \lim_{a\to\infty} \int_1^a \left(\frac{1}{x} + \frac{-1x}{x^2+1}\right) dx$$

$u = x^2+1$

$$= \lim_{a\to\infty} \ln|x| - \frac{1}{2}\ln(x^2+1) \Big|_1^a$$

$$= \lim_{a\to\infty} \underline{\ln a - \frac{1}{2}\ln(a^2+1)} - \ln 1 + \frac{1}{2}\ln 2$$

Properties of Ln 4.3

0

$$= \lim_{a\to\infty} \ln\left(\frac{a/a}{\sqrt{\frac{a^2}{a^2} + \frac{1}{a^2}}}\right) + \frac{1}{2}\ln 2$$

$\frac{1}{a^2} \to 0$

$$\ln 1 + \frac{1}{2}\ln 2 = \boxed{\frac{1}{2}\ln 2}$$ converges

$$\int_2^\infty \frac{\ln x}{x^2}\,dx =$$ Conv or div?

IBP

$u = \ln x \quad dv = x^{-2}\,dx$

$du = \frac{1}{x}\,dx \quad v = -x^{-1}$

$$= \lim_{a\to\infty}\left(-\frac{\ln x}{x}\Big|_2^a + \int_2^a \underbrace{\frac{1}{x}\cdot\frac{1}{x}}^{\frac{1}{x^2}}\,dx \right)$$

$$= \lim_{a\to\infty} -\frac{\ln x}{x} - \frac{1}{x}\Big|_2^a$$

$$= \lim_{a\to\infty} \cancel{\frac{-\ln a}{a}}^{\,0} - \cancel{\frac{1}{a}}^{\,0} + \frac{\ln 2}{2} + \frac{1}{2}$$

L'Hospital's Rule $\quad \lim_{a\to\infty} -\frac{\ln a}{a} = \lim_{a\to\infty} \frac{-\frac{1}{a}}{1} = 0$

$$= \boxed{\frac{\ln 2}{2} + \frac{1}{2}}$$ converges