Final Exam Summary of New Material
(NOTE that the Final Exam is comprehensive!)

• M.1

- A finite stochastic process is a sequence of repeated experiments whose outcomes depend on previous results.

- A Markov process is a finite stochastic process whereby the probabilities of each experiment depend ONLY on the prior result.

- A transition matrix is a matrix \( T \) whose entries are given by \( T_{ij} = P(\text{row } i | \text{column } j \text{ prior}) \). A matrix \( T \) is a transition matrix if all the following hold:
  * \( T \) is a square matrix
  * \( 0 \leq T_{ij} \leq 1 \) for all entries
  * Each column sums to 1

- The initial state \( X_0 \) of a Markov process is a column matrix whose entries are the probabilities of each outcome initially. Subsequent states can be found by the formula \( X_n = T^n X_0 \).

• M.2

- If the columns of a transition matrix \( T \) approach the same limiting value when raised to a large power, \( T \) is called a regular stochastic matrix.

- The process with such a matrix is called a regular Markov process.

- The limiting value of the columns is called the steady-state solution \( (X_S: \text{does NOT depend on an initial state}) \)

- \( T \) is a regular stochastic matrix if \( (T^n)_{ij} > 0 \) for SOME power \( n \) and all \( ij \) entries. (i.e., some power of \( T \) has no zeros)

- Equations which can be used to find the steady-state solution \( X_S \):

\[
TX_S = X_S \\
x_S + y_S + \cdots = 1
\]

• M.3

- An Absorbing State of a Markov Process is a state which, once entered, it is impossible to leave.

- A Markov Process is an Absorbing Markov Process if:
  * There is at least one “1” on the diagonal
  * There is at least one other nonzero entry in the row containing the “1”.

- If \( T \) is the transition matrix of an absorbing Markov process, the rows and columns can be rearranged to the form

\[
T = \begin{bmatrix}
I_{a \times a} & A_{a \times b} \\
0_{b \times a} & 0_{b \times b}
\end{bmatrix}
\]

If the exponent \( n \) gets large, \( T^n \rightarrow \begin{bmatrix}
I_{a \times a} & A(I-B)^{-1} \\
0_{b \times a} & 0_{b \times b}
\end{bmatrix} \) (this is called the limiting matrix)

• G.1

- A game is a situation of conflict between two parties.

- The payoff matrix of a game is a matrix \( A \) such that \( A_{ij} \) is the result for player 1 (“row”) when they choose strategy \( i \) and player 2 (“column”) chooses strategy \( j \). Player 2’s result is \( K - A_{ij} \), where \( K \) is some constant.

- An element in a payoff matrix which is the smallest element in the row and largest element in the column (if such an element exists) is a saddle point.

- If a payoff matrix has a saddle point, the game is a strictly determined game, the saddle point is the value of the game, and the row and column represent the optimal strategies of each player.

- Use the m&M strategy to locate a saddle point:
  1. Place an \( m \) next to the minimum of each row.
  2. Place an \( M \) next to the maximum of each column.
  3. Any entry with both an \( m \) and an \( M \) next to it is a saddle point.

Final Exam New Material Examples in Class

• M.1

1. Since 1852 (the current 2-party system), when a Democrat was president, 47% of the next elections were won by Democrats. When a Republican was president, 60% of the next elections were won by Republicans. Illustrate this with a tree diagram and determine the probability a Republican will win the 2024 Presidential election.

2. The following are the results of a (made up) survey of Northgate sandwich shop habits among Aggies:

   - Among Aggies who ate at Potbelly last time, 65% ate at Potbelly again, 35% ate at Subway.
   - Among Aggies who ate at Subway last time, 20% ate at Potbelly, 50% ate at Subway again, and 30% ate at Schlotzsky’s.

   Suppose 20% of those surveyed ate at Potbelly, 30% ate at Subway, and 50% ate at Schlotzsky’s. What are the probabilities an Aggie surveyed will eat at each of the three sandwich shops the next time?
M.2

1. Since 1852 (the current 2-party system), when a Democrat was president, 47% of the next elections were won by Democrats. When a Republican was president, 60% of the next elections were won by Republicans. Write the equations needed to find the steady-state solution, then find the steady-state using any method.

2. The following are the results of a (made up) survey of Northgate sandwich shop habits among Aggies:
   Among Aggies who ate at Potbelly last time, 65% ate at Potbelly again, 35% ate at Subway.
   Among Aggies who ate at Subway last time, 20% ate at Potbelly, 50% ate at Subway again, and 30% ate at Schlotzsky’s.
   Among Aggies who ate at Schlotzsky’s last time, 20% ate at Potbelly, 30% ate at Subway, and 50% ate at Schlotzsky’s.
   Suppose 20% of those surveyed ate at Potbelly, 30% ate at Subway, and 50% ate at Schlotzsky’s.
   Find the steady-state solution or explain why none exists.

3. Find the steady-state solution of the following transition matrix or explain why none exists. \( T = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \)

M.3

1. Given the transition matrix \( T = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \), explain why it is an absorbing matrix and find the limiting matrix.

2. A gambler’s game begins as follows: There are two pairs of envelopes: one pair contains two $10 bills in each envelope, the other pair consists of one envelope with $10 and one with $30. Player 1 chooses which pair of envelopes they will play with; Player 2 then chooses which envelope they get.
   The second part of the game consists of a sequence of coin tosses. If heads is tossed, Player 1 wins $10 from Player 2; otherwise, Player 2 wins $10 from Player 1. The game ends when one player runs out of money.
   (a) Write the transition matrix for the second part of the game. Determine whether it is a regular or absorbing stochastic matrix.
   (b) What happens when you find the steady-state for this matrix?
   (c) Use the results of the 100th coin toss to estimate the probability of each player winning from each of the possible starting points.
   (d) Find the limiting matrix.

3. In the transition matrix below, \( L = \{\text{lower division Business major}\}, \quad D = \{\text{drop out of Business}\}, \quad G = \{\text{graduated with Business degree}\}\):
   \[
   T = \begin{bmatrix}
   D & G & L & U \\
   D & 1 & 0 & .2 & .02 \\
   G & 0 & 1 & 0 & .33 \\
   L & 0 & 0 & .5 & 0 \\
   U & 0 & 0 & 0 & .3 & .65 
   \end{bmatrix}
   \]
   Find the probability a student starting in Business (lower division) will eventually graduate with a Business degree.

G.1

1. Given the payoff matrix below, decide if the game is strictly determined. If so, find the value of the game and the optimal strategy for each player:
   \[
   \begin{bmatrix}
   4 & -4 & -2 & 0 \\
   2 & 0 & -1 & 0 \\
   9 & 3 & -3 & -2 
   \end{bmatrix}
   \]
   (a) Write the payoff matrix for this game.
   (b) Determine if the game is Strictly Determined. If so, find the value of the game and the optimal strategy for each.

2. Each player has a penny, nickel, and dime. On the count of 3, they each reveal one of their coins. If the coins are the same, no one wins; otherwise, the player with the higher-valued coin wins the other player’s coin.
   (a) Write the payoff matrix for this game.
   (b) Determine if the game is Strictly Determined. If so, find the value of the game and the optimal strategy for each.

3. Rock beats Scissors, Paper beats Rock, and Scissors beats Paper. The winner receives the following: $1 if they played Rock, $5 if they played Paper, and $10 if they played Scissors.
   (a) Write the payoff matrix for this game.
   (b) Determine if the game is Strictly Determined. If so, find the value of the game and the optimal strategy for each.