COMPACTNESS OF HANKEL OPERATORS WITH CONTINUOUS SYMBOLS ON
CONVEX DOMAINS

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ABSTRACT. Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$, $n \geq 2$, $1 \leq q \leq (n-1)$, and $\phi \in C(\overline{\Omega})$. If the Hankel operator $H_{q-1}^{n-1}$ on $(0, q-1)$–forms with symbol $\phi$ is compact, then $\phi$ is holomorphic along $q$–dimensional analytic (actually, affine) varieties in the boundary. We also prove a partial converse: if the boundary contains only ‘finitely many’ varieties, $1 \leq q \leq n$, and $\phi \in C(\overline{\Omega})$ is analytic along the ones of dimension $q$ (or higher), then $H_{q-1}^{n-1}$ is compact.

1. INTRODUCTION AND RESULTS

Hankel operators on Bergman spaces on bounded pseudoconvex domains with symbols that are continuous on the closure of the domain are compact when the $\overline{\partial}$–Neumann operator on the domain is compact ([FS01, Has14, Str10]). It is natural to ask what happens when the $\overline{\partial}$–Neumann operator is not compact. Must there necessarily be noncompact Hankel operators (with, say, symbol continuous on the closure of the domain)? The answer is known only in cases where compactness (or lack thereof) of the $\overline{\partial}$–Neumann operator is understood: when the domain is convex, bounded, in $\mathbb{C}^n$ ([FS01], Remark 2), or when it is a smooth bounded pseudoconvex Hartogs domain in $\mathbb{C}^2$ ([SZ17]). In these cases, compactness of all Hankel operators with symbols that are smooth on the closure implies compactness of the $\overline{\partial}$–Neumann operator. In general, the question is open\(^1\); having an answer would be very interesting. Alternatively, one can consider a particular situation where the $\overline{\partial}$–Neumann operator is not compact and ask for necessary and/or sufficient conditions on a symbol that will imply compactness of the associated Hankel operator. Given that two symbols whose difference extends continuously to the boundary as zero yield Hankel operators that agree modulo a compact operator, one would in particular like to understand how the interaction of the symbol with the boundary affects compactness of the associated Hankel operator. It is this question that we are interested in in the current paper.

Such a study was initiated in [ÇŞ09b]. For symbols smooth on the closure of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, the authors show, under the condition that the rank of the Levi form

\(^{1}\)For the case $q = 0$ and $N_1$. The answer is known to be affirmative for Hankel operators on $(0, q)$–forms with $1 \leq q \leq (n-1)$ ([ÇŞ14]); the relevant $\overline{\partial}$–Neumann operator is then $N_{q+1}$. Here, pseudoconvexity is important; see [ÇŞ12].
is at least \((n - 2)\), that compactness of the Hankel operator requires that the symbol is holomorphic along analytic discs in the boundary. When the domain is also convex, they can dispense with the condition on the Levi form.\(^2\) Moreover, for (smooth) convex domains in \(\mathbb{C}^2\), holomorphy of the symbol along analytic discs in the boundary is also sufficient for compactness of the Hankel operator. Further contributions are in [Le10, ĆŞ17, CŞ18, Clo20]; we refer the reader to the introduction in [CČŞ18] for a summary. The latter authors significantly reduce the regularity requirements on both the domain (Lipschitz in \(\mathbb{C}^2\) or convex in \(\mathbb{C}^n\)) and the symbol (in \(C(\overline{\Omega})\)) that is required to infer holomorphicity of the symbol along analytic discs in the boundary from compactness of the Hankel operator. Because compactness of a Hankel operator localizes ([CČŞ18]; see also [Şah12]), the latter result carries over to locally convexifiable domains. In [ČŞş20], the authors, among other things, extend the result from [CČŞ18] on convex domains in \(\mathbb{C}^n\) to Hankel operators on \((\overline{\partial})\)-closed \((0,q)\)-forms with \(0 \leq q \leq (n - 1)\), but with symbol assumed in \(C^1\) of the closure. When the (convex) domain is smooth and satisfies so called maximal estimates\(^3\), holomorphicity of the symbol along \((n - 1)\)-dimensional analytic (equivalently: affine) polydiscs in the boundary suffices for compactness of the associated Hankel operator.\(^4\) Finally we mention the recent [CJW20], where the authors consider Hankel operators with form symbols (replacing multiplication with the wedge product) and prove many of the results discussed above for this situation.

Our first result (Theorem 1) reduces the regularity of the symbol in Theorem 1 in [ČŞş20] to \(C(\overline{\Omega})\). It also corrects a geometric issue in the proof of Theorem 2 in [CČŞ18] (where the result is given for \(q = 1\) only).

Since necessary and sufficient conditions for compactness of the \(\overline{\partial}\)-Neumann operator are understood on convex domains ([FS98, FS01, Str10]), it is reasonable to expect that when the domain is convex, one should likewise obtain simple necessary and sufficient conditions on the symbol that will guarantee compactness of the Hankel operator. That is, one expects the converse of Theorem 1 to hold: when the symbol is holomorphic along \(q\)-dimensional varieties in the boundary, \(H^1_{\overline{\partial}}\) should be compact. As mentioned above, this implication is known for \(C^1\) convex domains in \(\mathbb{C}^2\) and symbol in \(C^1(\overline{\Omega})\) ([ĆŞ09b, Theorem 3], [CČŞ18, Theorem (Ćučković–Şahutoğlu)], as well as for some classes of Reinhardt domains in \(\mathbb{C}^2\), with symbol only assumed in \(C(\overline{\Omega})\) ([CŞ18, Theorem 1], [Clo20, Theorem 1]). Theorem 2 says that the implication is true for convex domains in higher dimensions as well, when there are only ‘finitely many’ varieties in the boundary. We

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\(^2\)In both these cases, compactness of the \(\overline{\partial}\)-Neumann operator was known to fail when there are discs in the boundary; [ŞS06, Theorem 1], [FS98, Theorem 1.1].

\(^3\)This condition is equivalent to a comparable eigenvalues condition for the Levi form of the boundary, see the discussion in [ČŞş20] and their references.

\(^4\)As noted in [ČŞş20], these two results combined imply that a convex domain that satisfies maximal estimates for \((0,q)\)-forms does not have any analytic varieties of dimension \(\geq q\) in its boundary except ones in top dimension \((n - 1)\) (and their subvarieties). It would be desirable to have a direct proof for this fact.
expect this additional assumption to be an artifact of our current proof. On the other hand, our result appears to be the first instance of a converse to Theorem 1 in dimension greater than two.

Before stating our results precisely, we recall the notation form [CŚS20] (which is fairly standard). For a bounded domain $\Omega \subset \mathbb{C}^n$, denote by $K^2_{(0,q)}(\Omega)$ the space of square integrable $\overline{\partial}$–closed $(0,q)$–forms, and by $A^2_{(0,q)}(\Omega)$ the subspace of forms with holomorphic coefficients. $P_q : L^2_{(0,q)}(\Omega) \rightarrow K^2_{(0,q)}(\Omega)$ is the orthogonal projection, the Bergman projection on $(0,q)$–forms. For a symbol $\phi \in L^\infty(\Omega)$, the associated Hankel operator is $H^q_\phi : K^2_{(0,q)}(\Omega) \rightarrow L^2_{(0,q)}(\Omega)$,

$$H^q_\phi f = \phi f - P_q(\phi f),$$

with $\|H^q_\phi\| \leq \|\phi\|_{L^\infty(\Omega)}$. $H^q_\phi$ equals the commutator $[\phi, P_q]$ (since $P_q f = f$), so that statements about (compactness of) Hankel operators may also be viewed as statements about commutators between the Bergman projection and multiplication operators. When the symbol $\phi$ is in $C^1(\Omega)$, Kohn’s formula, $P_q = \overline{\partial}^* N_{q+1} \overline{\partial}$, implies

$$(1) \quad H^q_\phi f = \overline{\partial}^* N_{q+1}(\overline{\partial}\phi \wedge f);$$

here, $N_{q+1}$ is the $\overline{\partial}$–Neumann operator on $(0,q+1)$–forms.

We can now state our first result.

**Theorem 1.** Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$, $n \geq 2$, and $\phi \in C(\overline{\Omega})$. Let $\psi : \mathbb{D}^q \rightarrow b\Omega$ be a holomorphic embedding for some $q$ with $1 \leq q \leq n - 1$. If $H^{q-1}_\phi$ is compact on $A^2_{(0,q-1)}(\Omega)$ (a fortiori if it is compact on $K^2_{(0,q-1)}(\Omega)$), then $\phi \circ \psi$ is holomorphic.

Before stating Theorem 2, we first recall the basic facts about analytic varieties in the boundaries of convex domains from [FS98, Section 2 and Proposition 3.2] and [ČŚ09b, Lemma 2]. Suppose $\psi : \mathbb{D}^q \rightarrow b\Omega$ is a holomorphic embedding, where $\Omega \subset \mathbb{C}^n$ is convex. Then the convex hull of the image $\psi(\mathbb{D}^q)$ is contained in the intersection of a complex hyperplane $H$ through $P \in \psi(\mathbb{D}^q)$ with $\overline{\Omega}$ ([FS98, Section 2]). So it suffices to consider affine varieties in the boundary of this kind. Among these varieties through a boundary point $P$, there is a unique one, denoted by $V_p$, that has $P$ as a relative interior point and whose dimension $m$ is maximal. Then $0 \leq m \leq (n - 1)$, and the relative closure $\overline{V_p}$ is the intersection of an $m$–dimensional affine subspace through $P$ (contained in $H$) with $\overline{\Omega}$.

Our second result is as follows.

**Theorem 2.** Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n, \phi \in C(\overline{\Omega})$, and $1 \leq q \leq n$. Assume that the boundary of $\Omega$ contains at most finitely many disjoint varieties $\{\overline{V}_{P_k}\}$ of dimension $q$ or higher as above, $k = 1, \ldots, N$. Furthermore, assume that $\phi \circ \psi$ is holomorphic for every embedding $\psi : \mathbb{D}^q \rightarrow b\Omega$. Then $H^{q-1}_\phi$ is compact on $K^2_{(0,q-1)}(\Omega)$.

Note that the condition on holomorphicity of $\phi \circ \psi$ just says that $\phi|_{V_{P_k}}$ is holomorphic on all $V_{P_k}$ of dimension $q$ or higher. We also point out that this assumption on $\phi$ does not imply that the
tangential component of $\partial \phi$ (say when $\Omega$ is smooth) vanishes on the $V_{P_k}$ of dimension $q$ or higher; the components transverse to the varieties need not be zero.

If $\Omega$ is assumed $C^1$, the relative closures of distinct varieties are automatically disjoint: if $Q \in V_{P_{k_1}} \cap V_{P_{k_2}}$, then both $V_{P_{k_1}}$ and $V_{P_{k_2}}$ have to be contained in the complex tangent space to $b\Omega$ at $Q$, and considering the convex hull as above produces a variety which contains them both. However, without a regularity assumption on $b\Omega$, the supporting complex hyperplane is not unique, and two distinct varieties may share a boundary point (as on the boundary of $D \times D$).

When $q = n$, there are no $q$–dimensional varieties in the boundary, and the theorem says that $H_{n-1}^\phi$ is always (when $\phi \in C(\overline{\Omega})$) compact. But this is clear from (1), at least when $\phi \in C^1(\overline{\Omega})$, because $N_n$, and hence $\overline{\partial}^* N_n$, is compact ([FS98, Theorem 1.1]). When $\phi$ is merely in $C(\overline{\Omega})$, it can be approximated uniformly on $\Omega$ by smooth functions; the corresponding (compact) Hankel operators converge in norm to $H_{n-1}^\phi$.

As pointed out in [CÇS18, Remark 1], Theorem 2 fails on general domains, that is, without the assumption of convexity or a related condition. This failure is related to the subtleties surrounding compactness in the $\overline{\partial}$–Neumann problem on general domains (but absent in the case of convex domains). Namely, there are smooth bounded pseudoconvex complete Hartogs domains in $C^2$ without discs in the boundary and noncompact Hankel operators on them whose symbols are smooth on the closure. These symbols trivially satisfy the assumption on holomorphicity along analytic discs. The domains were originally constructed in [Mat98] as examples of smooth bounded pseudoconvex complete Hartogs domains without discs in the boundary whose $\overline{\partial}$–Neumann operator $N_1$ is nevertheless not compact (see also [FS01, Theorem 10], [Str10, Theorem 4.25]). But on these domains, noncompactness of $N_1$ implies that there are symbols smooth on the closure so that the associated Hankel operator is not compact ([SZ17, Theorem 1]).

2. Proofs

The following simple lemma formulates the lack of holomorphicity of a function without relying on differentiability, in contrast to [ÇSS20]; this ultimately allows the regularity of the symbol to be lowered from $C^1(\overline{\Omega})$ to $C(\overline{\Omega})$ in Theorem 1.

**Lemma 1.** Let $\Omega$ be a domain in $C^n$ and $\phi \in C(\Omega)$ that is not holomorphic. Then there exist $h \in C^\infty_0(\Omega)$ and $1 \leq j \leq n$ such that $\int_\Omega \phi \overline{h z_j} \neq 0$.

**Proof.** Assume that $\int_\Omega \phi \overline{h z_j} = 0$ for all $j$ and $h \in C^\infty_0(\Omega)$. This means that $\overline{\partial} \phi = 0$ as a distribution on $\Omega$. But $\overline{\partial}$ is elliptic in the interior, so that $\phi$ is an ordinary holomorphic function, a contradiction. \[\square\]

**Proof of Theorem 1.** We start the proof of Theorem 1 as in [ÇSS20, Proof of Theorem 1]. After dilation, translation and rotation if necessary, we assume that $(2D)^d \times \{0\} \subset b\Omega$ and, seeking to

\[\text{5Alternatively, } \overline{\partial} \phi = 0 \text{ implies } \Delta \phi = 0, \text{ so by Weyl’s Lemma, } \phi \text{ is a } C^\infty \text{ function (and therefore holomorphic).} \]
derive a contradiction, that the function \( \phi(z_1,\ldots,z_q,0,\ldots,0) \) is not holomorphic on \( \mathbb{D}^q \). Let us denote \( \Omega_0 = \{(z_{q+1},\ldots,z_n) \in \mathbb{C}^{n-q} : (0,\ldots,0,z_{q+1},\ldots,z_n) \in \Omega\} \), the slice transversal to \( \mathbb{D}^q \) through the origin. Then convexity of \( \Omega \) implies that \( \mathbb{D}^q \times \beta \Omega_0 \subset \Omega \) for any \( 0 < \beta \leq 1/2 \) ([FS98, page 636, proof of (1) \( \Rightarrow \) (2)]; [CCS18, Proof of Theorem 2]). Lemma 1 implies that there exist \( h \in C_0^\infty(\mathbb{D}^q) \) and \( \delta > 0 \) such that

\[
\left| \int_{\mathbb{D}^q} \phi(z_1,\ldots,z_q,0,\ldots,0) \frac{\partial h(z_1,\ldots,z_q)}{\partial z_1} dV(z_1,\ldots,z_q) \right| \geq 2\delta.
\]

Furthermore, continuity of \( \phi \) implies that there exists \( 0 < \beta < 1/2 \) such that

\[
\left| \int_{\mathbb{D}^q} \phi(z_1,\ldots,z_n) \frac{\partial h(z_1,\ldots,z_q)}{\partial z_1} dV(z_1,\ldots,z_q) \right| \geq \delta
\]

for all \((z_{q+1},\ldots,z_n) \in \beta \Omega_0\). As in [CCS18, Proof of Theorem 2] we use their Lemma 5 to produce a bounded sequence \( \{f_j\}_{j=1}^\infty = \{f_j(z_n)\}_{j=1}^\infty \subset A^2(\Omega) \) such that \( \|f_j\|_{L^2(\beta \Omega_0)} = 1 \) and \( f_j \to 0 \) weakly in \( A^2_{(\beta,0)}(\Omega) \). We define \( \alpha_j = f_j d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_q \). Then the sequence \( \{\alpha_j\}_{j=1}^\infty \) also has (the analogues of) these two properties.

We have

\[
\delta^2 = \delta^2 \|f_j\|_{L^2(\beta \Omega_0)}^2 \leq \int_{\beta \Omega_0} \left| \int_{\mathbb{D}^q} \phi(z_1,\ldots,z_n) \frac{\partial h(z_1,\ldots,z_q)}{\partial z_1} dV(z_1,\ldots,z_q) \right|^2 f_j(z_n)\bar{f}_j(z_n) dV(z_{q+1},\ldots,z_n)
\]

\[
= \int_{\beta \Omega_0} \left| \int_{\mathbb{D}^q} \phi(z_1,\ldots,z_n) f_j(z_n) \frac{\partial h(z_1,\ldots,z_q)}{\partial z_1} dV(z_1,\ldots,z_q) \right|^2 dV(z_{q+1},\ldots,z_n)
\]

\[
= \int_{\beta \Omega_0} \left| \int_{\mathbb{D}^q} \left( \phi \alpha_j, \frac{\partial h}{\partial \bar{z}_1} d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_q \right)_{\mathbb{D}^q} \right|^2 dV(z_{q+1},\ldots,z_n)
\]

\[
(2) = \int_{\beta \Omega_0} \left| \int_{\mathbb{D}^q} \left( \phi \alpha_j, \bar{\partial}_{z_1,\ldots,z_q} (h d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q) \right)_{\mathbb{D}^q} \right|^2 dV(z_{q+1},\ldots,z_n).
\]

In the last two lines \((\cdot,\cdot)\) denotes the standard inner product between forms on the fibers \( \mathbb{D}^q \times \{(z_{q+1},\ldots,z_n)\} \), and \( \bar{\partial}_{z_1,\ldots,z_q} \) is the adjoint of \( \partial_{z_1,\ldots,z_q} \), the \( \partial \) on each fiber (note that \( h d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \in dom(\bar{\partial}_{z_1,\ldots,z_q}) \) since \( h \in C_0^\infty(\mathbb{D}^q) \)). In the last equality above we used the fact that all the additional terms in \( \bar{\partial}_{z_1,\ldots,z_q} (h d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q) \) involve \( d\bar{z}_1 \) and are thus (even pointwise) orthogonal to \( \phi \alpha_j \).

Next, note that

\[
\bar{\partial}_{z_1,\ldots,z_q} (P_{q-1} \phi \alpha_j)_{\mathbb{D}^q} = (\bar{\partial}_{z_1,\ldots,z_n} P_{q-1} \phi \alpha_j)_{\mathbb{D}^q} = 0,
\]

\footnote{Alternatively, one can modify the argument below slightly and use the construction in [FS98, Proof of (1) \( \Rightarrow \) (2) in Theorem 1.1].}
where \(|_{\mathbb{D}^q}\) denotes the restriction (pull back) of forms to each fiber \(\mathbb{D}^q \times \{ (z_{q+1}, \ldots, z_n) \}\). Therefore, since \(\phi \alpha_j = \phi \alpha_j|_{\mathbb{D}^q}\) on \(\mathbb{D}^q \times \beta \Omega_0\),

\[
\left( \phi \alpha_j, \bar{\partial}_z z_{1 \cdots q} h d \bar{z}_1 \wedge \cdots \wedge d \bar{z}_q \right)_{\mathbb{D}^q} = \left( (\phi \alpha_j - P_{q-1}(\phi \alpha_j))|_{\mathbb{D}^q}, \bar{\partial}_z z_{1 \cdots q} h d \bar{z}_1 \wedge \cdots \wedge d \bar{z}_q \right)_{\mathbb{D}^q} = \left( (H_{\phi}^{q-1} \alpha_j)|_{\mathbb{D}^q}, \bar{\partial}_z z_{1 \cdots q} h d \bar{z}_1 \wedge \cdots \wedge d \bar{z}_q \right)_{\mathbb{D}^q}
\]

and consequently

\[
\left| \left( \phi \alpha_j, \bar{\partial}_z z_{1 \cdots q} h d \bar{z}_1 \wedge \cdots \wedge d \bar{z}_q \right)_{\mathbb{D}^q} \right| \leq \left\| (H_{\phi}^{q-1} \alpha_j)|_{\mathbb{D}^q} \right\|_{L^2(\mathbb{D}^q)} \left\| \bar{\partial}_z z_{1 \cdots q} h d \bar{z}_1 \wedge \cdots \wedge d \bar{z}_q \right\|_{L^2(\mathbb{D}^q)}.
\]

Combining this estimate with (2) and observing that \(|_{\mathbb{D}^q}\) decreases norms (pointwise on each fiber \(\mathbb{D}^q\): the omitted terms containing \(d \bar{z}_m\) with \(m > q\) are orthogonal to \((H_{\phi}^{q-1} \alpha_j)|_{\mathbb{D}^q}\) and that \(\mathbb{D}^q \times \beta \Omega_0 \subseteq \Omega\)

\[
\delta \lesssim \left\| (H_{\phi}^{q-1} \alpha_j)|_{L^2(\Omega)} \right\|_{\mathbb{D}^q} \left\| \bar{\partial}_z z_{1 \cdots q} h d \bar{z}_1 \wedge \cdots \wedge d \bar{z}_q \right\|_{L^2(\mathbb{D}^q)}.
\]

Therefore, \(\{H_{\phi}^{q-1} \alpha_j\}_{j=1}^{\infty}\) does not converge to 0 in \(L^2_{(0,q-1)}(\Omega)\) as \(j \to \infty\) (since the second factor on the right hand side is nonzero).

On the other hand, the sequence \(\{\alpha_j\}_{j=1}^{\infty}\) converges to zero weakly in \(A_{(0,q-1)}^2(\Omega)\), and therefore in \(K_{(0,q-1)}^2(\Omega)\), as \(j \to \infty\), and compactness of \(H_{\phi}^{q-1}\) then forces convergence of \(\{H_{\phi}^{q-1} \alpha_j\}_{j=1}^{\infty}\) to zero in \(L^2_{(0,q-1)}(\Omega)\). This contradiction completes the proof of Theorem 1.

The above proof of Theorem 1 uses ideas from [FS98, ÇŞ09b, ÇÇS18, ÇŞS20]; in turn, these ideas can be traced back at least to [Cat81, DP81].

In the proof of Theorem 2, we will repeatedly use the following sufficient condition for compactness of an operator \(T : X \to Y\), where \(X\) and \(Y\) are Hilbert spaces (see e.g. [Str10, Lemma 4.3(ii))]: for all \(\varepsilon > 0\) there is a Hilbert space \(Z_{\varepsilon}\), a linear compact operator \(S_{\varepsilon} : X \to Z_{\varepsilon}\), and a constant \(C_{\varepsilon}\) such that

\[
\|Tx\|_Y \leq \varepsilon \|x\|_X + C_{\varepsilon}\|S_{\varepsilon}x\|_{Z_{\varepsilon}}.
\]

In addition, we need the following sufficient conditions for compactness of a Hankel operator on \(\Omega\) (notation as in the theorem).

**Lemma 2.** Suppose that \(\Omega\) is as in Theorem 2 and

(i) \(\phi \in C^1(\overline{\Omega})\) and \(\bar{\partial}\phi\) vanishes on \(\bigcup_{j=1}^{N} V_{P_j}\), or

(ii) \(\phi \in C(\overline{\Omega})\) vanishes on \(\bigcup_{j=1}^{N} V_{P_j}\).

Then \(H_{\phi}^{q-1}\) is compact on \(K_{(0,q-1)}^2(\Omega)\).

Note that the condition in (i) is stronger than saying that \(\phi\) is holomorphic along the \(V_{P_j}\): \(\bar{\partial}\phi = 0\) also in the directions transverse to \(V_{P_j}\).
Proof. We start with (i)\(^7\). We want to estimate \(\|H_{\phi}^{q-1}f\|^2\) in such a way that we can use (3). So fix \(\varepsilon > 0\). In view of (1), we have

\[
\|H_{\phi}^{q-1}f\|^2 = \langle \bar{\phi}^* N_q(\bar{\phi} \wedge f), \bar{\phi}^* N_q(\bar{\phi} \wedge f) \rangle = \langle N_q(\bar{\phi} \wedge f), \bar{\phi} \wedge f \rangle
\]

for \(f \in K_{(0,q-1)}^2(\Omega)\). The (absolute value of the) inner product on the right hand side is

\[
\sum_{j=1}^n \sum'_{|K|=q-1} \int_{\Omega} (N_q(\bar{\phi} \wedge f))_{jK}(\bar{\phi}/\bar{z}_j) f_{jK} dV \leq \sum_{j=1}^n \|\bar{\phi}/\bar{z}_j\| N_q(\bar{\phi} \wedge f) \|f\|,
\]

with the usual notation \(f = \sum'_{|K|=q-1} f_{jK} d\bar{z}_j\), the \('\) indicating summation over increasing multi–indices, and \(jK = (j,k_1,\ldots,k_{q-1})\). Using the inequality \(2ab \leq (a^2/\varepsilon + \varepsilon b^2)\), where \(a, b > 0\), gives

\[
2\|\bar{\phi}/\bar{z}_j\| N_q(\bar{\phi} \wedge f) \|f\| \leq \frac{1}{\varepsilon} \|\bar{\phi}/\bar{z}_j\| N_q(\bar{\phi} \wedge f) \|f\|^2 + \varepsilon\|f\|^2.
\]

Because \((\bar{\phi}/\bar{z}_j)\) vanishes on \(\cup_{j=1}^N \overline{P_j}\), it is a compactness multiplier on \((0,q)\)–forms ([CS09a, Proposition 1, Theorem 3]): for all \(\varepsilon' > 0\), there is a constant \(C_{\varepsilon',j}\) such that

\[
\|\bar{\phi}/\bar{z}_j\| \leq \varepsilon'(\|\bar{\phi}\|^2 + \|\bar{\phi}^* u\|^2) + C_{\varepsilon',j} \|u\|^2_{-1},
\]

\(u \in dom(\bar{\phi}) \cap dom(\bar{\phi}^* \in L_{(0,q)}^2(\Omega)\). We apply (7) to the form \(u = N_q(\bar{\phi} \wedge f)\) on the right hand side of (6), with \(\varepsilon' = \varepsilon^2\), to obtain that the right hand side of (6) is dominated by

\[
\varepsilon\|\bar{\phi} N_q(\bar{\phi} \wedge f)\|^2 + C_{\varepsilon,j}\|N_q(\bar{\phi} \wedge f)\|^2_{-1} + \varepsilon\|f\|^2
\]

\[
\lesssim \varepsilon\|f\|^2 + C_{\varepsilon,j} \|N_q(\bar{\phi} \wedge f)\|^2_{-1};
\]

the constants involved (other than \(C_{\varepsilon,j}\)) are independent of \(\varepsilon\) and \(f\) (but not of \(\phi\)). We have used here that \(N_q(\bar{\phi} \wedge f) \in dom(\bar{\phi}) \cap dom(\bar{\phi}^*)\) and that \(\bar{\phi} N_q(\bar{\phi} \wedge f) = 0\) (since \(\bar{\phi} f = 0\)). Combining (4) –(6) and (8) and summing over \(j = 1,\ldots,n\) results in the estimate we are looking for:

\[
\|H_{\phi}^{q-1}f\|^2 \lesssim \varepsilon\|f\|^2 + C_{\varepsilon}\|N_q(\bar{\phi} \wedge f)\|^2_{-1}; f \in K_{(0,q-1)}^2(\Omega),
\]

with \(C_{\varepsilon} = \max\{C_{\varepsilon,j} | j = 1,\ldots,n\}\).

The operator \(f \mapsto N_q(\bar{\phi} \wedge f)\) is continuous from \(K_{(0,q-1)}^2(\Omega)\) to \(L_{(0,q)}^2(\Omega)\), hence is compact to \(W_{(0,q)}^{-1}(\Omega)\) (see e.g. ([Gri11, Theorem 1.4.3.2]: \(W_{0}^{1}(\Omega) \hookrightarrow L^2(\Omega)\) is compact; by duality, so is \(L^2(\Omega) \hookrightarrow W^{-1}(\Omega)\)). Therefore (3) is satisfied for \(T = H_{\phi}^{q-1}\), with \(X = K_{(0,q-1)}^2(\Omega)\), \(Y = L_{(0,q-1)}^2(\Omega)\), \(Z_\varepsilon = W_{(0,q)}^{-1}(\Omega)\), and \(S_\varepsilon = N_q(\bar{\phi} \wedge f) : K_{(0,q-1)}^2(\Omega) \rightarrow W_{(0,q)}^{-1}(\Omega)\), and \(H_{\phi}^{q-1}\) is compact. (In this particular case, \(Z_\varepsilon\) and \(S_\varepsilon\) are independent of \(\varepsilon\).)

\((ii)\) is an easy consequence of \((i)\). Any symbol as in \((ii)\) can be approximated uniformly on \(\overline{\Omega}\) by symbols in \(C^1(\overline{\Omega})\) and vanishing in a neighborhood of the compact set \(\cup_{j=1}^N \overline{P_j}\). These symbols

\(^7\)When \(\Omega\) is smooth, \((i)\) is a special case of Theorem 1.3 in [CJW20].
satisfy the assumption in (i), and the corresponding (compact) Hankel operators converge to \( H^{q-1}_\phi \) in norm\(^8\). Therefore, \( H^{q-1}_\phi \) is compact as well. \(^9\)

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** For the moment, fix \( j, 1 \leq j \leq N \). \( \mathcal{V}_p \) is a convex subset of an affine subspace, and \( \phi \) is holomorphic on \( \mathcal{V}_p \) and continuous on the closure. Via dilatation, we can approximate \( \phi \) uniformly of \( \mathcal{V}_p' \), first by functions holomorphic in a relative neighborhood of \( \mathcal{V}_p' \), and then (by trivial extension) holomorphic in a neighborhood in \( C^n \) of \( \mathcal{V}_p' \).

To prove compactness of \( H^{q-1}_\phi \), we use the sufficient condition in (3). So fix \( \epsilon > 0 \). For \( j = 1, \ldots, N \), choose open sets \( U_j \) in \( C^n \) that contain \( \mathcal{V}_p \) and are pairwise disjoint, and cutoff functions \( \chi_j \in C^\infty_0(U_j), \) \( 0 \leq \chi_j \leq 1 \), which are identically equal to one in a neighborhood of \( \mathcal{V}_p' \). In light of the previous paragraph, we can do that in such a way that there exists a holomorphic function \( h_j \) on \( U_j \) that first approximates \( \phi \) to within \( \epsilon \) on \( \mathcal{V}_p' \), and then by continuity to within \( 2\epsilon \) on \( U_j \cap \overline{\Omega} \) (shrinking \( U_j \) if necessary). Also set \( \chi_0 = (1 - \sum_{j=1}^N \chi_j) \in C(\overline{\Omega}) \). With this setup, we split \( H^{q-1}_\phi \) as follows:

\[
H^{q-1}_\phi = H^{q-1}_{(\chi_0 \phi)} + H^{q-1}_{(\sum_{j=1}^N \chi_j \phi)} = H^{q-1}_{(\chi_0 \phi)} + H^{q-1}_{(\sum_{j=1}^N \chi_j h_j)} + H^{q-1}_{(\sum_{j=1}^N \chi_j (\phi - h_j))}.
\]

\( \chi_0 \phi \) vanishes on \( \bigcup_{j=1}^N \mathcal{V}_p \), and Lemma 2, part (ii), implies that \( H^{q-1}_{(\chi_0 \phi)} \) is compact. It remains to consider the remaining terms on the right hand side of (10).

We have \( \sum_{j=1}^N \chi_j h_j \in C^\infty(\overline{\Omega}) \) and \( \partial(\sum_{j=1}^N \chi_j h_j) = \sum_{j=1}^N h_j \partial \chi_j \). This symbol therefore satisfies the assumptions in part (i) of Lemma 2, and the corresponding Hankel operator is compact. Because \( \|\chi_j(\phi - h_j)\|_{L^\infty(\Omega)} \leq 2\epsilon \), and the \( \chi_j \) have disjoint supports, the norm of the last operator in (10) is bounded by \( 2\epsilon \), and we conclude

\[
\|H^{q-1}_{\phi} f\| \leq 2\epsilon \|f\| + \|H^{q-1}_{(\chi_0 \phi + \sum_{j=1}^N \chi_j h_j)} f\|.
\]

Estimate (11) is (3) (after rescaling \( \epsilon \)) with \( X = K^2_{(0,q-1)}(\Omega), Y = L^2_{(0,q-1)}(\Omega), Z_\epsilon = L^2_{(0,q-1)}(\Omega), \) and the compact operator \( S_\epsilon = H^{q-1}_{(\chi_0 \phi + \sum_{j=1}^N \chi_j h_j)} = H^{q-1}_{\chi_0 \phi} + H^{q-1}_{\chi_j h_j} \) (in contrast to \( Z_\epsilon, S_\epsilon \) does depend on \( \epsilon \), via \( \chi_j \) and \( h_j \)). As \( \epsilon > 0 \) was arbitrary, we conclude that \( H^{q-1}_\phi \) is compact and complete the proof of Theorem 2. \( \square \)

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\(^8\)Because \( \|H^{q-1}_\phi - H^{q-1}_{\phi'}\| = \|H^{q-1}_{\phi - \phi'}\| \leq \|\phi - \phi'\|_{L^\infty(\Omega)} \).

\(^9\)By [CS09a, Proposition 1, Theorem 3], \( \phi \) is a compactness multiplier for \((0,q)-\)forms. When \( \Omega \) is smooth (and \( q = 1) \), one can use [CZ16, Theorem 1]: symbols which are compactness multipliers on a bounded smooth pseudoconvex domain produce compact Hankel operators.
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