CHAPTER 14 – APPORTIONMENT

An ***apportionment problem*** is to round a set of fractions so their sum is maintained at its original value.

The rounding procedure used in an apportionment problem is called an ***apportionment method***.

Notation
- Round q to the nearest integer is $[q]$. Round half integers up.
- Round q down is $\lfloor q \rfloor$
- Round q up is $\lceil q \rceil$

The total population, p, divided by the house size, h, is called the ***standard divisor, s***.

$$s = \frac{p}{h}$$

A group's quota q_i is the group's population, p_i, divided by the standard divisor, s.

$$q_i = \frac{p_i}{s}$$

Hamilton Method

1. Round each quota down.
2. Calculate the number of seats left to be assigned.
3. Assign the seats to those with the largest fractional parts.

Use the Hamilton method to apportion 36 silver coins to Doris, Mildred, and Henrietta if Doris paid \$5900, Mildred paid \$7600, and Henrietta paid \$1400.

$S = \frac{14900}{36} = 413.8889$

		Quota	$\lfloor q \rfloor$	
Doris	5900	5900/413.8889 = 14.2550	14	14
Mildred	7600	= 18.3624	18	18
Henrietta	1400	= 3.3825	3 +1 = 4	4
	\$14,900		35	36 ✓

When the bag of coins is opened, there are 37 coins. How should they be apportioned?

$S = \frac{14900}{37} = 402.7027$

		Quota	$\lfloor q \rfloor$	
Doris	5900	5900/402.7027 = 14.6510	14 +1 = 15	15
Mildred	7600	= 18.8725	18 +1 = 19	19
Henrietta	1400	= 3.4765	3	3
			35	37 ✓

A ***paradox*** is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.

The ***Alabama paradox*** occurs when a state loses a seat as the result of an increase in the house size.

The ***new states paradox*** occurs in a reapportionment in which an increase in the total number of seats causes a shift in the apportionment of existing states.

The ***population paradox*** occurs when there are a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

Consider two numbers, A and B, where $A > B$.

The ***absolute difference*** between the two numbers is $A - B$

The ***relative difference*** between the two numbers is $\frac{A-B}{B} \times 100\%$

Find the absolute and relative difference between the following numbers

(a) 1 and 2 ABS: $2-1=1$, REL: $\frac{2-1}{1} \times 100\% = 100\%$

(b) 10 and 11 ABS: $11-10=1$, REL: $\frac{11-10}{10} \times 100\% = 10\%$

(c) 2000 and 2001 ABS: $2001-2000=1$, REL: $\frac{2001-2000}{2000} \times 100\% = 0.05\%$

A county has 4 districts, North, South, East, and West. They will apportion for a 100 member advisory council using the method of Hamilton. Determine the number of council members from each district.

$S = 64920/100 = 649.2$

		quota	$\lfloor q \rfloor$	
North	27,460	$27{,}460/649.2 = 42.2982$	42	42
South	17,250	$= 26.5712$	26 + 1	= 27
East	19,210	$= 29.5903$	29 + 1 =	30
West	1000	$= 1.5404$	1	1
	64920		98	100 ✓

Ten years later the county will reapportion for the 100 seats using new population data. Determine the number of council members from each district. Did a paradox occur? $S = 65910/100 = 659.1$

old			q	$\lfloor q \rfloor$	% Change
42	North	28,140	42.69	42 + 1 = 43	$(28140-27460)/27460 \times 100\% = 2.476\%$
27	South	17,450	26.48	26 26	$(17450-17250)/17250 \times 100\% = 1.159\%$
30	East	19,330	29.33	29 29	$(19330-19210)/19210 \times 100\% = 0.625\%$
1	West	990	1.50	1 + 1 = 2	$990-1000/990 \times 100\% = -1.010\%$
		65,910		98	

The pop paradox occurred → West decreased but got an extra repr.

A country has two states, Solid and Liquid. Use Hamilton's method to apportion 12 seats for their congress.

$S = \frac{203995}{12} = 16999.5833$

		Quotas	⌊q⌋	
Solid	144,899	8.5237	8 +1 = 9	
Liquid	59,096	3.4763	3	3
	203,995		11	12 ✓

Another state, Plasma, wants to join. If there are 38,240 people in that state, how many representatives should they receive?

$\frac{38\,240}{16{,}999.5833} = 2.2495 \rightarrow 2$

Apportion using Hamilton's method. $12 + 2 = 14$ house size

$S = 242235/14 = 17302.5$

				⌊q⌋		
9	Solid	144,899	8.374	8	8	changed old ones ⇒ new state paradox occurred
3	Liquid	59,096	3.415	3 +1 = 4	4	
	~~Gas~~ Plasma	38,240	2.210	2	2	
	$p = 242{,}235$			13		

Did any paradoxes occur?

Divisor Methods

The standard divisor, s, represents the average district population. Apportionment can be done by adjusting the average district population to be a specific value called the ***adjusted divisor, d***.

A ***divisor method*** of apportionment determines each state's apportionment by dividing its population by a common divisor d and rounding the resulting quota. Different divisor methods use different rounding rules.

A ***critical divisor*** is a divisor that will produce a quota for each population that gives a correct total number of seats.

The Jefferson Method

1. Find the standard divisor, s. Then find $q_i = \left\lfloor \frac{p_i}{s} \right\rfloor$
2. If the total number of seats is not correct, find the new divisors that correspond to giving each state one more seat. $d_i = \frac{p_i}{q_i + 1}$
3. Assign a seat to the state with the largest d_i. If the total number of seats is correct, stop. Otherwise repeat step 2.
4. The adjusted divisor d will be the exact value of the last divisor found in step 3.

Use the Jefferson method to apportion 36 silver coins to Doris, Mildred, and Henrietta if Doris paid \$5900, Mildred paid \$7600, and Henrietta paid \$1400. Comment on your results.

$S = 14900/36 = 413.8889$ MOST FAIR

		Quota	$\lfloor q \rfloor$	d	
Doris	5900	14.2550	14	$5900/(14+1) = 393.33$	14
Mildred	7600	18.3624	18	$7600/(18+1) = 400 \Rightarrow$	19
Henrietta	1400	3.3825	3	$1400/(3+1) = 350$	3
			35		

When the bag of coins is opened, there are 37 coins. How should they be apportioned? Comment on your results

$S = 14900/37 = 402.7027$

$\frac{7600}{7600/19}$ d

				d
Doris	5900	14.65	14	$5900/(14+1) = 393.33$
Mildred	7600	18.87	18	$7600/(18+1) = 400 \Rightarrow$ new divisor, d
Henrietta	1400	3.48	3	$1400/(3+1) = 350$
			35	

$d = 400$

	Quotas	$\lfloor q \rfloor$			
D	$5900/400 = 14.75$	14	$5900/(14+1) = 393.33$	+1	15
M	$7600/400 = 19$	19	$7600/(19+1) = 380$		19
H	$1400/400 = 3.5$	3	$1400/(3+1) = 350$		3
		36			

A school offers four different art classes with the enrollments shown below. Ten new teachers will be hired using Jefferson's method. Determine who gets the new teachers and comment on the results.

$S = 1192/10 = 119.2$

		Quota	$\lfloor q \rfloor$	d
Ceramics	785	$785/119.2 = 6.59$	6	$785/(6+1) = 112.14$ +1 ⇒ $d = 785/7$
Painting	152	1.28	1	$152/(1+1) = 76$
Dance	160	1.34	1	$160/(1+1) = 80$
Theatre	95	0.80	0	$95/(0+1) = 95$
			8	

$p = 1192$

	Quota	$\lfloor q \rfloor$	d	d
C	$785/(785/7) = 7$	7	$785/(7+1) = 98.125$ +1 ⇒	8
P	$152/(785/7) = 1.36$	1	$152/(1+1) = 76$	1
D	$160/(785/7) = 1.43$	1	$160/(1+1) = 80$	1
T	$95/(785/7) = .85$	0	$95/(0+1) = 95$	0
		9		

→ violates the Quota rule

Quota Rule says that the number assigned to each represented unit must be the standard quota, q_i, rounded up or rounded down.

Balinski and Young found that no apportionment method that satisfies the quota rule is free of paradoxes.

The Adams Method

1. Find the standard divisor, s. Then find $N_i = \lceil q_i \rceil = \left\lceil \frac{p_i}{s} \right\rceil$
2. If the total number of seats is not correct, find the new divisors that correspond to giving each state one fewer seat. $d_i = \frac{p_i}{q_i - 1}$
3. Remove a seat from the state with the smallest d_i. If the total number of seats is correct, stop. Otherwise repeat step 2.
4. The adjusted divisor d will be the exact value of the last divisor found in step 3.

The Webster Method

1. Find the standard divisor, s. Then find $N_i = [q_i] = \left[\frac{p_i}{s} \right]$
2. If the total number of seats is correct, the process is done.
3. If the total number of seats is too few, use a critical divisor of d^+ and the state with the largest critical divisor gets a next seat

$$d_i^+ = \frac{p_i}{N_i + \frac{1}{2}}$$

4. If the total number of seats is too many, use a critical divisor of d^- and the state with the smallest critical divisor loses a seat

$$d_i^- = \frac{p_i}{N_i - \frac{1}{2}}$$

Apportion the regions below using both the Adams method and the Webster method for a house size of 16. Comment on your results.

$S = \frac{45136}{16} = 2821$

		Quota	initial $\lceil q \rceil$	d	new divisor	
Beach	28,204	9.998	10	28204/(10−1) = 3134	loses 1	9
Forest	11,267	3.99	4	11267/(4−1) = 3756		4
Plains	4203	1.49	2	4203/(2−1) = 4203		2
Swamp	1462	.52	1	1462/(1−1) = BIG#		1
	45136		17			16 ✓

Webster

			[q]
Beach	28,204	9.998	10
Forest	11,267	3.99	4
Plains	4203	1.49	1
Swamp	1462	.52	1
			16 ✓

Geometric Mean

The arithmetic mean of two numbers a and b is given by

$$\bar{x} = \frac{a+b}{2}$$

The ***geometric mean*** of two numbers a and b is given by

$$G(a,b) = \sqrt{ab}$$

Example

Find the arithmetic and geometric means for the following numbers.

(a) 0 and 1.

$\bar{x} = \frac{0+1}{2} = .5$

$G = \sqrt{0 \times 1} = 0$

(b) 4 and 5

$\bar{x} = \frac{4+5}{2} = 4.5$

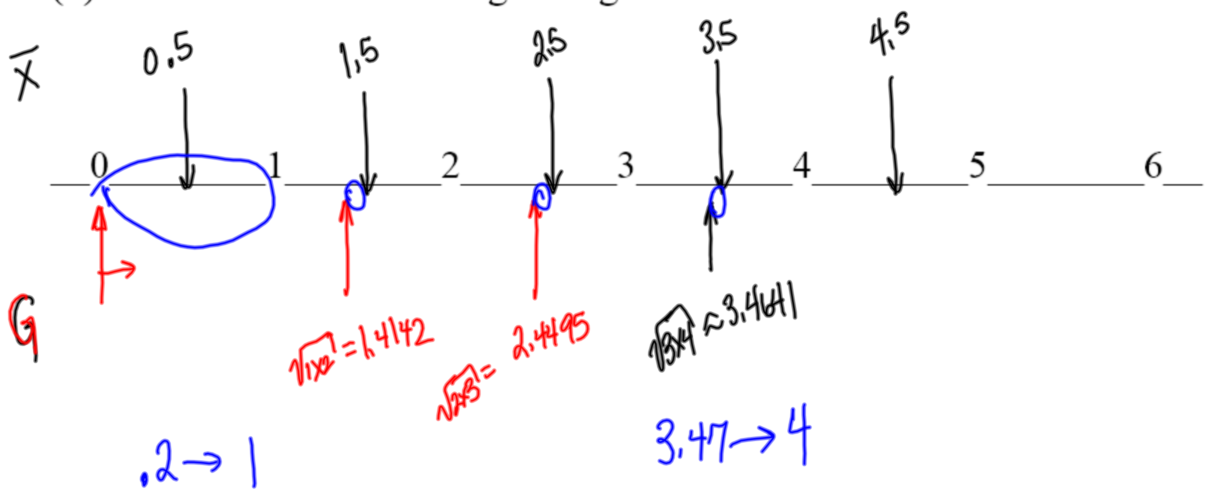
$G = \sqrt{4 \times 5} \approx 4.4721$

(c) 25 and 26

$\bar{x} = \frac{25+26}{2} = 25.5$

$G = \sqrt{25 \times 26} = 25.4951$

(d) Round 0.2 and 3.47 using their geometric means.



The Hill-Huntington Method

1. Find the standard divisor, s. Then find $q_i = \frac{p_i}{s}$
2. Round each quota q_i up or down by comparing it to $q_i^* = \sqrt{\lfloor q_i \rfloor \times \lceil q_i \rceil}$
3. If the total number of seats is correct, the process is done.
4. If the total number of seats is too few, implement Jefferson's method with a critical divisor of $d_i^+ = \frac{p_i}{\sqrt{N \times (N+1)}}$
5. If the total number of seats is too many, implement Adams' method with a critical divisor of $d_i^- = \frac{p_i}{\sqrt{N \times (N-1)}}$

Apportion the regions below using the Hill-Huntington method for a house size of 16.

$S = 2821$

		Quota	q^*			
Beach	28,204	9.998	$\sqrt{9\times10} = 9.48$	10	$28204/\sqrt{10(10-1)} = 2973$	10
Forest	11,267	3.99	$\sqrt{3\times4} = 3.46$	4	$11267/\sqrt{4(4-1)} = 3253$	4
Plains	4203	1.49	$\sqrt{1\times2} = 1.41$	2	$4203/\sqrt{2(2-1)} = 2972$ lose 1	1
Swamp	1462	0.52	$\sqrt{0\times1} = 0$	1	$1462/\sqrt{1(1-1)} =$ BIG #	1
				17		16 ✓

A country has 5 states and a house size of 40. Apportion for their congress using the following methods: Hamilton, Jefferson, Adams, Webster and Hill-Huntington.

Northwest	130,451	10.3219	
Southwest	157,633	12.4767	
Central	25,879	2.0477	
Southeast	88,956	7.0386	
Northeast	102,611	8.1191	

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SAMPLE EXAM QUESTIONS FROM CHAPTER 14

1. Match the definition to the term

(A) Quota Rule is III

(B) Alabama Paradox is II

(C) Population Paradox is IV

I. An increase in the total number of seats causes a shift in the apportionment of existing states.

II. A state loses a seat as the result of an increase in the house size.

III. The number assigned to each represented unit must be the standard quota, q_i, rounded up or rounded down.

IV. A reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

2. Round the following numbers using the geometric mean. $\sqrt{3 \times 4} = 3.4641$

0.39 rounds to 1 3.47 rounds to 4

3. Use the Webster method of apportionment to distribute 12 seats on a city council to three districts with the populations shown below.

$S = \frac{75000}{12} = 6250$

District	Population	Quota	[q]	d	final
North	30,000	30000/6250 = 4.8	5	30000/5-½ = 6667	5
Central	10,100	1.616	2	10100/2-½ = 6733	2
South	34,900	5.584	6	34900/6-½ = 6345 lose 1	5
	75000		13		12 ✓

4. A school is offering Gothic literature, Hispanic literature and Women writers. There are enough teachers to teach 10 sections of literature. We know that 58 people signed up for G, 20 for H and 42 for W.

(a) Use Hamilton's method to determine the number of sections of each class that will be taught.

(b) Use Jefferson's method to determine the number of sections of each class that will be taught.

$S = 120/10 = 12$

		Q	$\lfloor q \rfloor$		final
G	58	$58/12 = 4.83$	4	+1	5
H	20	$20/12 = 1.67$	1	+1	2
W	42	$42/12 = 3.5$	3		3
	120		8		10 ✓

Jeff

$S = 12$

2 short

	d new divisor	Quota	$\lfloor q \rfloor$	new new divisor	final
G	$58/(4+1) = 11.6$	$58/11.6 = 5$	5	$58/(5+1) = 9.67$	5
H	$20/(1+1) = 10$	$20/11.6 = 1.72$	1	$20/(1+1) = 10$	1
W	$42/(3+1) = 10.5$	$42/11.6 = 3.62$	3	$42/(3+1) = 10.5$ new divisor	4
			9		