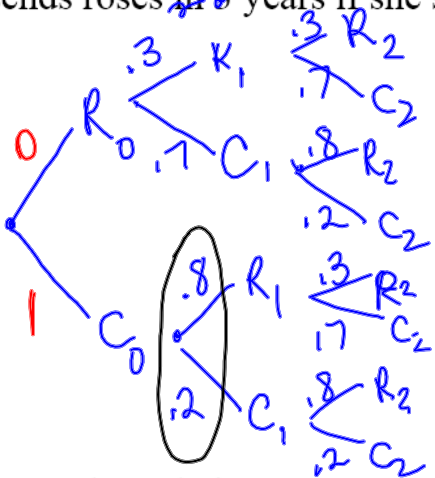
Markov Chains

Every Mother's Day, a certain individual sends her mother either roses or carnations. If she sends roses one year, 30% of the time she will send roses again the next year. If she sends carnations one year, 80% of the time she will send roses the next year. What is the probability that she sends roses ~~in~~ 2 ~~5~~ years if she sent carnations this year?



$P(R_2) = .8 \times .3 + .2 \times .8 = .4$ cum

$$X_0 = \begin{array}{c} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}\begin{array}{c} R \\ C \end{array}, \quad T = \begin{array}{c} \\ R \\ C \end{array}\begin{array}{c} \begin{array}{cc} R & C \end{array} \\ \begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix} \end{array}$$

next

$$TX_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix} = X_1$$

$[A][B]$

$$X_2 = TX_1 = T(TX_0) = T^2 X_0$$

A Markov chain or process describes an experiment consisting of a finite number of stages.

- The outcomes and associated probabilities at each stage depend only on the outcomes of the preceding stage.
- The outcome at any stage of the experiment in a Markov chain is called the state of the experiment.

A transition matrix T is a matrix such that:

- The matrix is square
- All entries are nonnegative.
- The entries in each column sum to 1.
- The entries represent conditional probabilities

The initial state is stored as matrix X_0. The matrix X_i represents the distribution after i stages.

$$X_n = T^n X_0$$

Example

Every Mother's Day, a certain individual sends her mother either roses or carnations. If she sends roses one year, 30% of the time she will send roses again the next year. If she sends carnations one year, 80% of the time she will send roses the next year.

a) Find the transition matrix.
b) If there is a 40% chance of giving roses this year, what is the probability that she sends roses in ten years?
c) Given she sent carnations this year, what is the probability that she will give carnations again 10 years from now?

a) $\begin{array}{c|cc} & R & C \\ \hline R & .3 & .8 \\ C & .7 & .2 \end{array}$

b) $X_0 = \begin{array}{c} R \\ C \end{array}\begin{bmatrix} .4 \\ .6 \end{bmatrix}$ [B] $\quad X_{10} = T^{10}\begin{bmatrix} .4 \\ .6 \end{bmatrix} = \begin{bmatrix} .5332 \\ .4668 \end{bmatrix}$ $= [A]^\wedge 10 * [B]$ P(R)

c) $X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X_{10} = T^{10}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .5328 \\ .4672 \end{bmatrix}$ P(C)

The steady state (long term) distribution of is X_L and $TX_L = X_L$.

Example

What is the long term distribution for flowers on Mother's Day?

$X_L = \begin{bmatrix} x \\ y \end{bmatrix}, \quad TX_L = X_L \Rightarrow \begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} .3x + .8y \\ .7x + .2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \left.\begin{array}{l} .3x + .8y = x \\ .7x + .2y = y \end{array}\right\} \to$ move x,y to LHS

$\boxed{\begin{array}{l} -.7x + .8y = 0 \\ .7x - .8y = 0 \end{array}}$

$\boxed{x + y = 1}$

$\Rightarrow \left[\begin{array}{cc|c} -.7 & .8 & 0 \\ .7 & -.8 & 0 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 8/15 \\ 0 & 1 & 7/15 \\ 0 & 0 & 0 \end{array}\right]$

In the long run she gives roses 8/15th of the time and carnations 7/15th of the time.

Example

A study has shown that a family living in the state of Denial typically takes a vacation once per year. The vacations can be in-state, out-of-state or international. The transition matrix is

$$\begin{array}{cc} & \begin{array}{ccc} IS & OS & IT \end{array} \\ \begin{array}{c} IS \\ T = OS \\ IT \end{array} & \begin{bmatrix} 0.10 & 0.20 & 0.60 \\ 0.50 & 0.25 & 0.35 \\ 0.40 & 0.55 & 0.05 \end{bmatrix} \end{array}$$

$$X_L = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ and $x + y + z = 1$

What is the long term distribution of vacation destinations?

$$T X_L = X_L$$

$$\begin{bmatrix} .1 & .2 & .6 \\ .5 & .25 & .35 \\ .4 & .55 & .05 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{array}{l} .1x + .2y + .6z = x \quad -x \\ .5x + .25y + .35z = y \quad -y \\ .4x + .55y + .05z = z \quad -z \end{array}$$

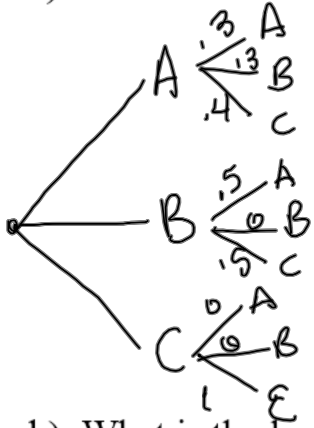
$$\begin{array}{l} -.9x + .2y + .6z = 0 \\ .5x - .75y + .35z = 0 \\ .4x + .55y - .95z = 0 \\ x + y + z = 1 \end{array} \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 52/171 \\ 0 & 1 & 0 & 41/114 \\ 0 & 0 & 1 & 115/342 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \approx .3041 \\ \approx .3596 \\ \approx .3363 \\ \end{array}$$

In the long term, 52/171 of the travel is in state, 41/114 is out of state and 115/342 is international

Example

A company offers three different cars to its executives each year. Those who have a brand A car ask for a brand A car again 30% of the time, they ask for a brand B car 30% of the time and a brand C car 40% of the time. Those who are driving a brand B car ask for a brand A car 50% of the time and a brand C car 50% of the time. Those who are driving a brand C car ask for a brand C car all of the time.

a) Find the transition matrix for this Markov process.



$$T = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} .3 & .5 & 0 \\ .3 & 0 & 0 \\ .4 & .5 & 1 \end{bmatrix} \end{array}$$

b) What is the long term distribution of cars?

All drive brand C

A transition matrix T is a ***regular*** Markov chain if the sequence T, T^2, T^3,… approaches a steady state matrix with all positive entries.
no zeros

An ***absorbing*** transition matrix has the following properties:

1. There is at least one absorbing state
2. It is possible to go from any non-absorbing state to an absorbing state in one or more stages.

An absorbing state is a unit column with the 1 on the main diagonal.

<u>Example</u>
Classify the following matrices as regular, absorbing, neither, or not a transition matrix.

$$\begin{bmatrix}0.9 & 0.2\\ 0.1 & 0.8\end{bmatrix}$$ Regular

$$\begin{bmatrix}0.7 & 1\\ 0.3 & 0\end{bmatrix}^2 = \begin{bmatrix}.79 & .7\\ .21 & .3\end{bmatrix} \Rightarrow \text{Regular}$$

$$\begin{bmatrix}1 & 0.5\\ 0 & 0.5\end{bmatrix}^{100} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \Rightarrow \text{absorbing}$$

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} .3 & 0 & 0 & 0\\ .1 & 1 & 0 & .5\\ .4 & 0 & 1 & 0\\ .2 & 0 & 0 & .5 \end{bmatrix} \end{array}$$

B and C are absorbing
this is an absorbing markov matrix

$$\begin{array}{c c} & \begin{array}{ccc} L & M & N \end{array} \\ \begin{array}{c} L \\ M \\ N \end{array} & \begin{bmatrix} .3 & .2 & 0\\ .7 & .8 & 0\\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

N is absorbing state
but not an absorbing markov matrix

neither

Absorbing stochastic matrices can be rewritten as $\left[\begin{array}{c|c} I & S \\ \hline 0 & R \end{array}\right]$

The steady state solution is $\begin{bmatrix} I & S(I-R)^{-1} \\ 0 & 0 \end{bmatrix}$

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1/8 & 0 & 0 \\ 0 & 5/8 & 0 & 0 \\ 0 & 0 & 1 & 1/6 \\ 0 & 1/4 & 0 & 5/6 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ A \\ C \\ B \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & C & B & D \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 1/8 & 0 \\ 0 & 1 & 0 & 1/6 \\ \hline 0 & 0 & 5/8 & 0 \\ 0 & 0 & 1/4 & 5/6 \end{array}\right] \end{array}$$

S R

$S = \begin{bmatrix} 1/8 & 0 \\ 0 & 1/6 \end{bmatrix} = [A]$ and $R = \begin{bmatrix} 5/8 & 0 \\ 1/4 & 5/6 \end{bmatrix} = [B]$ $\quad [A](\text{Identity}(2) - [B])^{-1}$

start

$$\begin{array}{c} \\ \text{end } A \\ \rightarrow C \\ B \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & C & B & D \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 2/3 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

If start in B, 1/3 chance you end up in A and 2/3 end up in C
If start in D, end up in C

The matrix $F = (I - R)^{-1}$ is called the fundamental matrix and the entry f_{ij} gives the expected number of times the system will be in the i^{th} nonabsorbing if it is initially in the j^{th} nonabsorbing state.

The sum of the entries in the j^{th} column of F is the expected number of stages before absorption if the system was initially in the j^{th} nonabsorbing state.

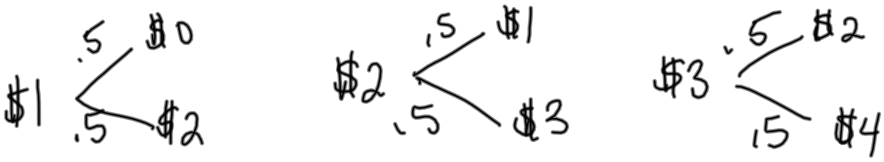
$$F = \left(\text{identity}(2) - [B]\right)^{-1} = \begin{array}{c c} & \begin{array}{cc} B & D \end{array} \\ \begin{array}{c} B \\ D \end{array} & \begin{pmatrix} 8/3 & 0 \\ 4 & 6 \end{pmatrix} \end{array}$$

If you start in B, you spend about $8/3 + 4 \approx 7$ turns before getting absorbed

If you start in D, you spend about 6 turns in D before absorbed

Example

A person plays a game in which the probability of winning \$1 is 0.50 and the probability of losing \$1 is 0.50. If she goes broke or reaches \$4, she quits. Find the long-term behavior if she starts with \$1, \$2, or \$3.



$$\begin{array}{c|ccccc} & \$0 & \$4 & \$1 & \$2 & \$3 \\ \hline \$0 & 1 & 0 & .5 & 0 & 0 \\ \$4 & 0 & 1 & 0 & 0 & .5 \\ \hline \$1 & 0 & 0 & 0 & .5 & 0 \\ \$2 & 0 & 0 & .5 & 0 & .5 \\ \$3 & 0 & 0 & 0 & .5 & 0 \end{array}$$

$$S = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 0 & .5 \end{bmatrix} = [A]$$

$$R = \begin{bmatrix} 0 & .5 & 0 \\ .5 & 0 & .5 \\ 0 & .5 & 0 \end{bmatrix} = [B]$$

$$\text{Long term: } S(I-R)^{-1} = \begin{array}{c} \\ \$0 \\ \$4 \end{array} \begin{array}{ccc} \$1 & \$2 & \$3 \\ \hline .75 & .5 & .25 \\ .25 & .5 & .75 \end{array}$$

$$F = (I-R)^{-1} = \begin{array}{c|ccc} & \$1 & \$2 & \$3 \\ \hline \$1 & 1.5 & 1 & .5 \\ \$2 & 1 & 2 & 1 \\ \$3 & .5 & 1 & 1.5 \end{array}$$

If you start with \$1, you play 3 turns before exiting
\$2 4 turns " "
\$3, 3